COURSE MATERIAL

IV Year B. Tech I- Semester

MECHANICAL ENGINEERING

AY: 2024-25

FINITE ELEMENT ANALYSIS

R20A0330

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MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MECHANICAL ENGINEERING

(Autonomous Institution-UGC, Govt. of India)
Secunderabad-500100,Telangana State, India.
www.mrcet.ac.in

CONTENTS

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

VISION

- ❖ To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become technology leaders of Indian vision of modern society.

MISSION

- ❖ To become a model institution in the fields of Engineering, Technology and Management.
- ❖ To impart holistic education to the students to render them as industry ready engineers.
- ❖ To ensure synchronization of MRCET ideologies with challenging demands of International Pioneering Organizations.

QUALITY POLICY

- ❖ To implement best practices in Teaching and Learning process for both UG and PG courses meticulously.
- ❖ To provide state of art infrastructure and expertise to impart quality education.
- ❖ To groom the students to become intellectually creative and professionally competitive.
- ❖ To channelize the activities and tune them in heights of commitment and sincerity, the requisites to claim the never - ending ladder of **SUCCESS** year after year.

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VISION

To become an innovative knowledge center in mechanical engineering through state-of-the-art teaching-learning and research practices, promoting creative thinking professionals.

MISSION

The Department of Mechanical Engineering is dedicated for transforming the students into highly competent Mechanical engineers to meet the needs of the industry, in a changing and challenging technical environment, by strongly focusing in the fundamentals of engineering sciences for achieving excellent results in their professional pursuits.

Quality Policy

- ✓ To pursuit global Standards of excellence in all our endeavors namely teaching, research and continuing education and to remain accountable in our core and support functions, through processes of self-evaluation and continuous improvement.
- ✓ To create a midst of excellence for imparting state of art education, industry-oriented training research in the field of technical education.

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Department of Mechanical Engineering

PROGRAM OUTCOMES

Engineering Graduates will be able to:

1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and teamwork**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

12.Life-long learning: Recognize the need for and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSOs)

PSO1 Ability to analyze, design and develop Mechanical systems to solve the Engineering problems by integrating thermal, design and manufacturing Domains.

PSO2 Ability to succeed in competitive examinations or to pursue higher studies or research.

PSO3 Ability to apply the learned Mechanical Engineering knowledge for the Development of society and self.

Program Educational Objectives (PEOs)

The Program Educational Objectives of the program offered by the department are broadly listed below:

PEO1: PREPARATION

To provide sound foundation in mathematical, scientific and engineering fundamentals necessary to analyze, formulate and solve engineering problems.

PEO2: CORE COMPETANCE

To provide thorough knowledge in Mechanical Engineering subjects including theoretical knowledge and practical training for preparing physical models pertaining to Thermodynamics, Hydraulics, Heat and Mass Transfer, Dynamics of Machinery, Jet Propulsion, Automobile Engineering, Element Analysis, Production Technology, Mechatronics etc.

PEO3: INVENTION, INNOVATION AND CREATIVITY

To make the students to design, experiment, analyze, interpret in the core field with the help of other inter disciplinary concepts wherever applicable.

PEO4: CAREER DEVELOPMENT

To inculcate the habit of lifelong learning for career development through successful completion of advanced degrees, professional development courses, industrial training etc.

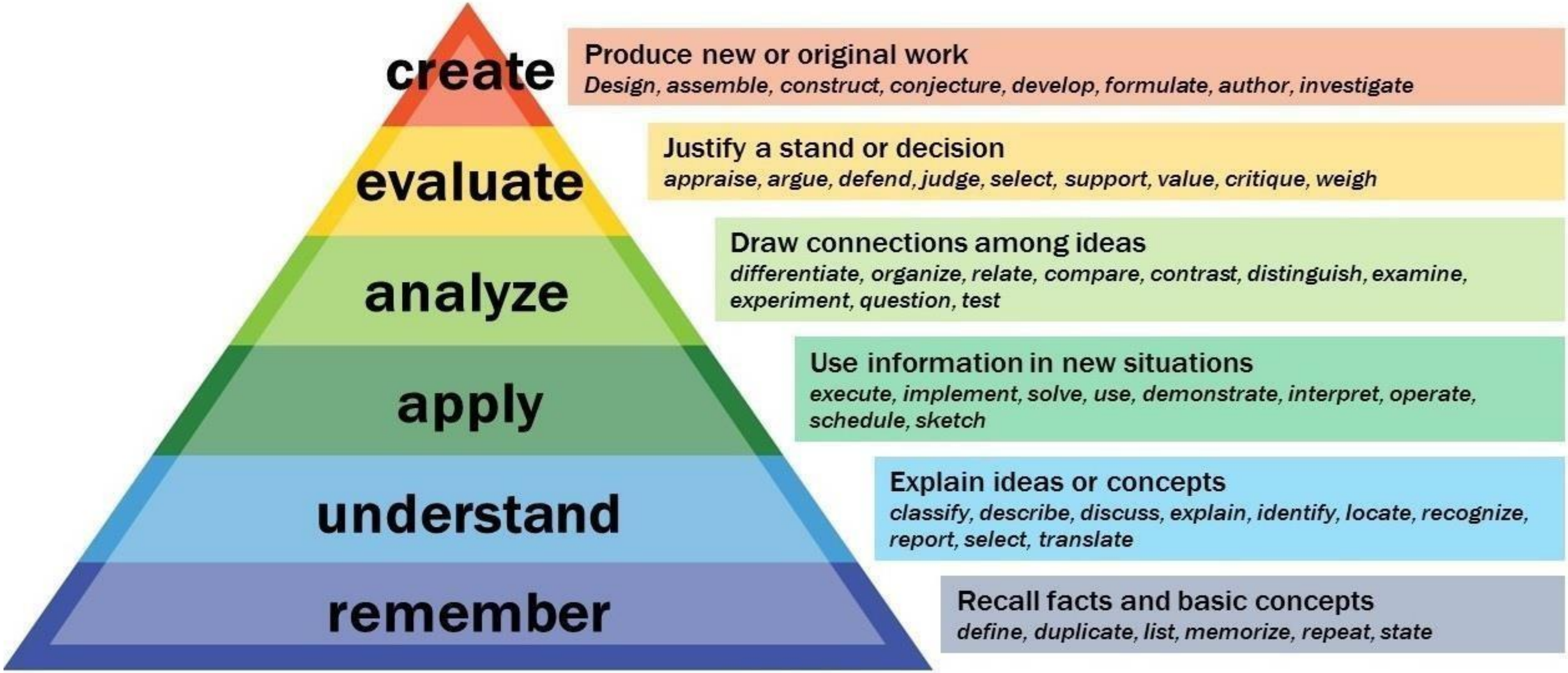
PEO5: PROFESSIONALISM

To impart technical knowledge, ethical values for professional development of the student to solve complex problems and to work in multi-disciplinary ambience, whose solutions lead to significant societal benefits.

Blooms Taxonomy

Bloom's Taxonomy is a classification of the different objectives and skills that educators set for their students (learning objectives). The terminology has been updated to include the following six levels of learning. These 6 levels can be used to structure the learning objectives, lessons, and assessments of a course.

1. **Remembering**: Retrieving, recognizing, and recalling relevant knowledge from long- term memory.
2. **Understanding**: Constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining.
3. **Applying**: Carrying out or using a procedure for executing or implementing.
4. **Analyzing**: Breaking material into constituent parts, determining how the parts relate to one another and to an overall structure or purpose through differentiating, organizing, and attributing.
5. **Evaluating**: Making judgments based on criteria and standard through checking and critiquing.
6. **Creating**: Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing.



create
Produce new or original work
Design, assemble, construct, conjecture, develop, formulate, author, investigate
evaluate
Justify a stand or decision
appraise, argue, defend, judge, select, support, value, critique, weigh
analyze
Draw connections among ideas
differentiate, organize, relate, compare, contrast, distinguish, examine, experiment, question, test
apply
Use information in new situations
execute, implement, solve, use, demonstrate, interpret, operate, schedule, sketch
understand
Explain ideas or concepts
classify, describe, discuss, explain, identify, locate, recognize, report, select, translate
remember
Recall facts and basic concepts
define, duplicate, list, memorize, repeat, state

Course Syllabus

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

IV Year B. Tech, ME-I Sem

L	T/P/D	C
3	0	3

(R20A0330) FINITE ELEMENT ANALYSIS

Objectives:

1. To enable the students to understand fundamentals of finite element analysis and theprinciple's involved in the discretization of domain with various elements, polynomial interpolation and assembly of global arrays.
2. To learn the application of FEM equations for trusses and Beams
3. To learn the application of FEM equations for axisymmetric problems and CST
4. To learn the application of FEM equations for Iso-Parametric and heat transfer problems
5. To learn the application of FEM equations for dynamic analysis

UNIT I

Introduction to Finite Element Method for solving field problems, Stress and Equilibrium, Strain - Displacement relations, Stress - strain relations.

One-Dimensional Problem: finite element modeling, local coordinates and shape functions. Potential Energy approach, Assembly of Global stiffness matrix and load vector. Finite element equations, Treatment of boundary conditions.

UNIT II

Trusses: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses.

BEAMS: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses.

Unit III

Two Dimensional Problems: Basic concepts of plane stress and plane strain, stiffness matrix of CST element, finite element solution of plane stress problems.

Axi-Symmetric Model: Finite element modelling of axi-symmetric solids subjected to axi-symmetric loading with triangular elements.

Unit IV

Iso-Parametric Formulation: Concepts, sub parametric, super parametric elements, 2 dimensional 4 nodded iso-parametric elements, and numerical integration.

Heat Transfer Problems: One dimensional steady state analysis composite wall. One dimensional fin analysis and two-dimensional analysis of thin plate.

Unit V

Dynamic Analysis: Formulation of finite element model, element matrices, evaluation of Eigen values and Eigen vectors for a stepped bar and a beam.

Text Books:

1. Tirupathi.R. Chandrupatla and Ashok D. Belegundu, Introduction to Finite elements in Engineering. PHI.
2. S Senthil, Introduction of Finite Element Analysis. Laxmi Publications.
3. SMD Jalaluddin, Introduction of Finite Element Analysis.Anuradha Publications.
4. The Finite Element Method for Engineers – Kenneth H. Huebner, Donald John Wiley & sons (ASIA) Pte Ltd.

References:

1. K. J. Bathe, Finite element procedures. PHI.
2. SS Rao, The finite element method in engineering. Butterworth Heinemann.
3. J.N. Reddy, An introduction to the Finite element method. TMH.
4. Chennakesava, R Alavala, Finite element methods: Basic concepts and applications. PHI.

Course Outcomes:

1. Identify mathematical model to solve common engineering problems by applying the finite element method and formulate the elements for one dimensional bar structures and solve problems in one dimensional bar structures.
2. Derive element matrices to find stresses in trusses and Beams
3. Formulate FEcharacteristic equations for axisymmetric problems and analyze plain stress,plain strain and Derive elementmatrices for CST elements.
4. Formulate FE Characteristic equations for Isoparamteric problems and heattransfer problem.
5. Solve dynamic problems where the effect of mass matters during the analysis.

Lecturer Notes

CONTENTS

COURSE COVERAGE SUMMARY

Units	Chapter No's In The Text Book Covered	Author	Text Book Title	Publishers	Edition
Unit-I Introduction to FEM and One dimesional Elements	1&2	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-II Trusses & Beams	3 &4	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-III Two dimensional Problems &Axi-symmetric Models	12	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-IV Iso-Parametric Formulation & Heat Transfer Problems	7&13	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-V Dynamic Analysis	14	S M Jalalud Ddin	Introduction of Finite Element Analysis	Anuradha Publications	4

UNIT 1

INTRODUCTION TO FEM

& ONE DIMENSIONAL PROBLEMS

Syllabus

Introduction to Finite Element Method for solving field problems. Stress and Equilibrium. Strain – Displacement relations. Stress – strain relations. One Dimensional problem: Finite element modeling, local coordinates and shape functions. Potential Energy approach, Assembly of Global stiffness matrix and load vector. Finite element equations, Treatment of boundary conditions, Quadratic shape functions and its applications.

OBJECTIVE:

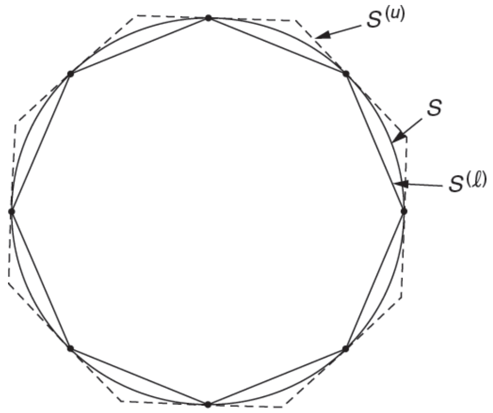
To enable the students to understand fundamentals of finite element analysis and the principle's involved in the discretization of domain with various elements, polynomial interpolation and assembly of global arrays. .

OUTCOME:

Identify mathematical model to solve common engineering problems by applying the finite element method and formulate the elements for one dimensional bar structures and solve problems in one dimensional bar structures.

Unit-I
Introduction to FEM

- Although the name of the finite element method was given recently, the concept dates back for several centuries. For example, ancient mathematicians found the circumference of a circle by approximating it by the perimeter of a polygon as shown in Figure
- In terms of the present day notation, each side of the polygon can be called a “finite element.” By considering the approximating polygon inscribed or circumscribed, one can obtain a lower bound $S^{(l)}$ or an upper bound $S^{(u)}$ for the true circumference S.



- Fig. Lower and Upper Bounds to the Circumference of a Circle.

- Furthermore, as the number of sides of the polygon is increased, the approximate values converge to the true value. These characteristics, as will be seen later, will holdtrue in any general finite element application.
- To find the differential equation of a surface of minimum area bounded by a specified closed curve, Schellback discretized the surface into several triangles and used a finite difference expression to find the total discretized area in 1851.
- In the current finite element method, a differential equation is solved by replacing it by a set of algebraic equations. Since the early 1900s, the behavior of structural frameworks, composed of several bars arranged in a regular pattern, has been approximated by that of an isotropic elastic body.
- In 1943, Courant presented a method of determining the torsional rigidity of a hollow shaft by dividing the cross section into several triangles and using a linear variation of the stress function ϕ over each triangle in terms of the values of ϕ at net points (called nodes in the present day finite element terminology).
- This work is considered by some to be the origin of the present-day finite element method. Since mid-1950s, engineers in aircraft industry have worked on developing approximate methods for the prediction of stresses induced in aircraft wings.
- In 1956, Turner, Cough, Martin, and Topp presented a method for modeling the wing

skin using three-node triangles. At about the same time, Argyris and Kelsey presented several papers outlining matrix procedures, which contained some of the finite element ideas, for the solution of structural analysis problems. Reference is considered as one of the key contributions in the development of the finite element method.

- The name finite element was coined, for the first time, by Clough in 1960. Although the finite element method was originally developed mostly based on intuition and physical argument, the method was recognized as a form of the classical Rayleigh- Ritz method in the early 1960s.
- Once the mathematical basis of the method was recognized, the developments of new finite elements for different types of problems and the popularity of the method started to grow almost exponentially.
- The digital computer provided a rapid means of performing the many calculations involved in the finite element analysis and made the method practically viable. Along with the development of high-speed digital computers, the application of the finite element method also progressed at a very impressive rate.
- Zienkiewicz and Cheung presented the broad interpretation of the method and its applicability to any general field problem. The book by Przemienieckipresents the finite element method as applied to the solution of stress analysis problems.

Definition:

- In Finite Element Analysis, the structure or body is divided into finite numbers of elements, the solution is obtained for individual element and solution of all elements is assembled to give distribution of field variable over entire region.

- For example: Heat Analysis $\longrightarrow$ Field variable is Temperature

  Stress & strain $\longrightarrow$ Field variable is Displacement.

Engineering Applications of the Finite Element Method

The finite element method was developed originally for the analysis of aircraft structures.

I. Equilibrium problems or steady-state or time-independent problems
   - In an equilibrium problem, we need to find the steady-state displacement or stress distribution if it is a solid mechanics problem,
   - Temperature or heat flux distribution if it is a heat transfer problem and
   - Pressure or velocity distribution if it is a fluid mechanics problem.

II. Eigen-value problems
   - In eigenvalue problems also, time will not appear explicitly. They may be considered as extensions of equilibrium problems in which critical values of certain parameters are to be determined in addition to the corresponding steady- state configurations.
   - In these problems, we need to find the natural frequencies or buckling loads and mode shapes if it is a solid mechanics or structures problem.

- Stability of laminar flows if it is a fluid mechanics problem and
- Resonance characteristics if it is an electrical circuit problem.

III. Propagation or transient problems

- The propagation or transient problems are time-dependent problems. This type of problemarises, for example, whenever we are interested in finding the response of a body undertime-varying force in the area of a solid mechanics
- Under sudden heating or cooling inthe field of heat transfer.
- Crack propagation.

Engineering Applications of the Finite Element Method

Area of Study	Equilibrium Problems	Eigenvalue Problems	Propagation Problems
1.Civil engineering structures	Static analysis of trusses, frames, folded plates, shell roofs, shear walls, bridges, and prestressed concrete structures	Natural frequencies and modes of structures; stability of structures	Propagation of stress waves; response of structures to a periodic loads
2. Aircraft structures	Static analysis of aircraft wings, fuselages, fins, rockets, spacecraft, and missile structures	Natural frequencies, flutter, and stability of aircraft, rocket, spacecraft, and missile structures	Response of aircraft structures to Random loads; dynamic response of aircraft and spacecraft to a periodic loads
3.Heat conduction	Steady-state temperature distribution in solids and fluids	–	Transient heat flow in rocket nozzles, internal combustion engines, Turbine blades, fins, and building structures
4. Geomechanics	Analysis of excavations, retaining walls, undergo und openings, rock joints, and soil-structure interaction problem; stress analysis in soils, dams, layered piles, and machine foundations	Natural frequencies and modes of dam-reservoir systems and soil-structure interaction problems	Time-dependent soil-structure interaction problems; transient seepage in soils and rocks; stress wave propagation in soils and rocks
5.Hydraulic and water resources engineering; hydrodynamics	Analysis of potential flows, free surface flows, boundary layer flows, viscous flows, transonic aerodynamic problems; analysis of hydraulic structures and dams	Natural periods and modes of shallow basins, lakes, and harbors; sloshing of liquids in rigid and flexible containers	Analysis of unsteady fluid flow and wave propagation problems; transient seepage in aquifers and porous media; rarefied gas dynamics ;magnetohydrodynamic flows

6. Nuclear engineering	Analysis of nuclear pressure vessels and containment structures; steady – State Temperature distribution in reactor components	Natural frequencies and stability of containment structure s; neutron Flux distribution	Response of reactor containment structures to dynamic loads; unsteady temperature distribution in reactor components; thermal and viscoelastic analysis of reactor structures
7. Biomedical engineering	Stress analysis of eyeballs, bones, and teeth; load-bearing capacity of implant and prosthetic systems; mechanics of heart valves	–	Impact analysis of skull; dynamics of anatomical structures
8. Mechanical Design	Stress concentration problems; stress analysis of pressure vessels, pistons, Composite materials, linkages, and gears	Natural frequencies and stability of linkages, gears, and machine tools	Crack and fracture problems under dynamic loads
9. Electricalmachine s and electromagnetics	Steady-state analysis of synchronous and induction machines, eddy current, and core loss in electric machines, magnetostatics	–	Transient behavior of Electromechanical devices such as motors and actuators, magnetodynamics

3D Elasticity:

EXTERNAL FORCES ACTING ON THE BODY

Two basic types of external forces act on a body

Body force (force per unit volume) e.g., weight, inertia, etc

Surface traction (force per unit surface area) e.g., friction

Strains: 6 independent **strain components**

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Consider the equilibrium of a differential volume element to obtain the 3 **equilibrium equations** of elasticity

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_a = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_c = 0$$

Compactly;

EQUILIBRIUM EQUATIONS	$\underline{\partial}^T \underline{\sigma} + \underline{X} = \underline{0}$	(1)

where

$$\underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

3D elasticity problem is completely defined once we understand the following three concepts

Strong formulation (governing differential equation + boundary conditions)

Strain-displacement relationship

Stress-strain relationship

Equilibrium equations	$\underline{\partial}^T \underline{\sigma} + \underline{X} = \underline{0} \quad in\ V$	(1)

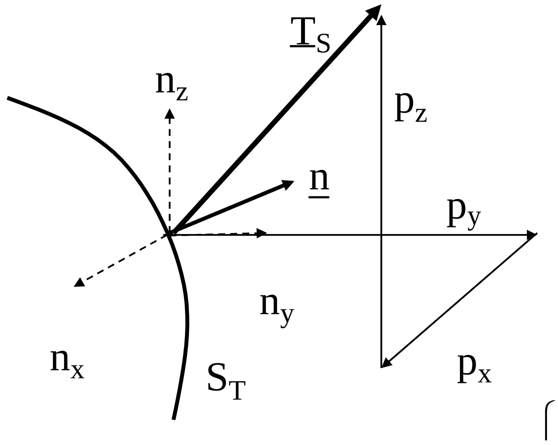
Boundary conditions

1. Displacement boundary conditions: Displacements are specified on portion S_u of the boundary

$$\boxed{\underline{u} = \underline{u}^{specified} \quad on \ \ S_u}$$

2. Traction (force) boundary conditions: **Tractions** are specified on portion S_T of the boundary

Now, how do I express this mathematically?



Traction: Distributed force **per unit area**

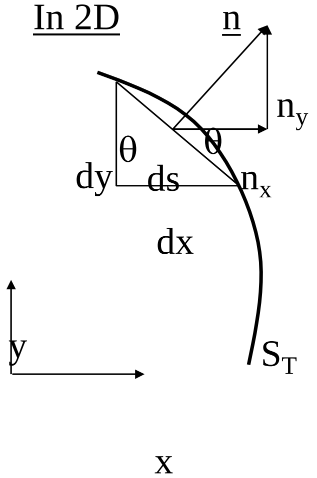
$$\underline{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

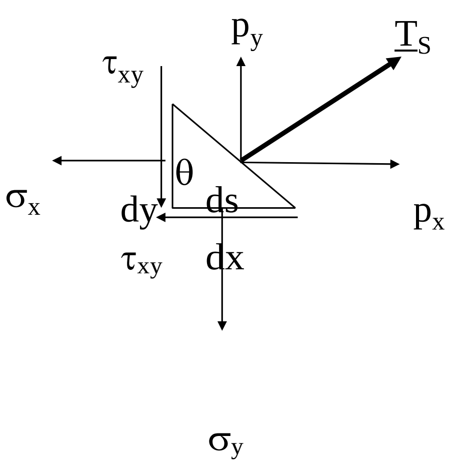
If the unit outward normal to S_T $\underline{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$

:Then

$$\boxed{\begin{aligned} p_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ p_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ p_z &= \tau_{xz} n_x + \tau_{zy} n_y + \sigma_z n_z \end{aligned}}$$

In 2D





Consider the equilibrium of the wedge in x-direction

$$\sin\theta = \frac{dx}{ds} = n_y$$

$$\cos\theta = \frac{dy}{ds} = n$$

$$p_x ds = \sigma_x dy + \tau_{xy} dx$$

$$\Rightarrow p_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$

$ds \quad x$

$$\Rightarrow p_x = \sigma_x n_x + \tau_{xy} n_y$$

Similarly

$$p_y = \tau_{xy} n_x + \sigma_y n_y$$

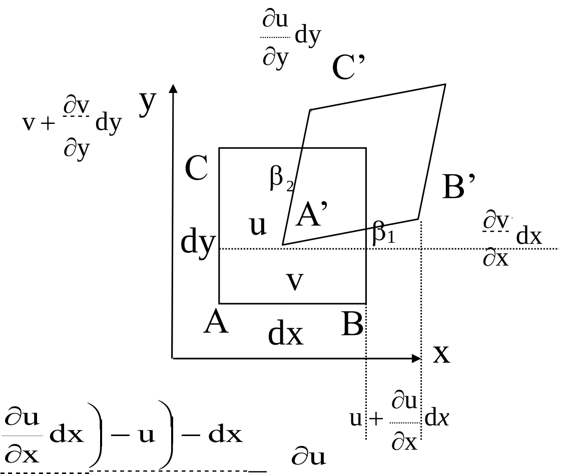
2. Strain-displacement relationships:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\varepsilon_z = \frac{\partial w}{\partial z}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$
$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Compactly; $\boxed{\underline{\varepsilon} = \underline{\partial}\,\underline{u}}$ (2)

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \qquad \underline{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

In 2D



$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\left(dx + \left(u + \frac{\partial u}{\partial x} dx\right) - u\right) - dx}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{A'C' - AC}{AC} = \frac{\left(dy + \left(v + \frac{\partial v}{\partial y} dy\right) - v\right) - dy}{dy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\pi}{2} - \text{angle } (C'A'B') = \beta_1 + \beta_2 \approx \tan\beta_1 + \tan\beta_2$$

$$\approx \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}$$

3. Stress-Strain relationship:

Linear elastic material (Hooke's Law)

$$\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}} \qquad (3)$$

Linear elastic isotropic material

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Special cases:

1. **1D elastic bar** (only 1 component of the stress (stress) is nonzero. All other stress (strain) components are zero)
Recall the (1) equilibrium, (2) strain-displacement and (3) stress-strain laws

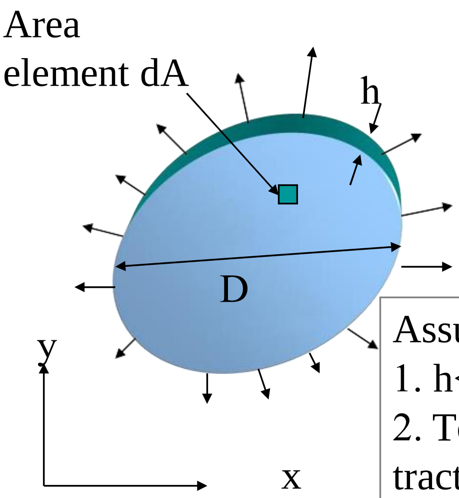
2. **2D elastic problems:** 2 situations

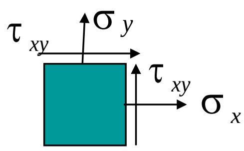
PLANE STRESS
PLANE STRAIN

3. 3D elastic problem: special case-**axisymmetric body with axisymmetric loading (we will skip this)**

PLANE STRESS: Only the in-plane stress components are nonzero

Nonzero stress components $\sigma_x, \sigma_y, \tau_{xy}$

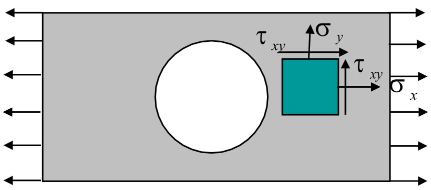




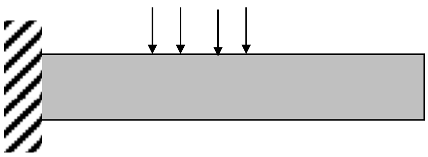
Assumptions:
1. h<<D
2. Top and bottom surfaces are free from traction
3. X_c=0 and p_z=0

PLANE STRESS Examples:

1. Thin plate with a hole



2. Thin cantilever plate



PLANE STRESS

Nonzero **stresses**: $\sigma_x, \sigma_y, \tau_{xy}$

Nonzero **strains**: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}$

Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

Hence, the $\underline{D}$ matrix for the **plane stress case** is

$$\underline{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

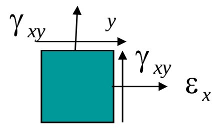
<u>PLANE STRAIN</u>: Only the in-plane <u>strain</u> components are nonzero

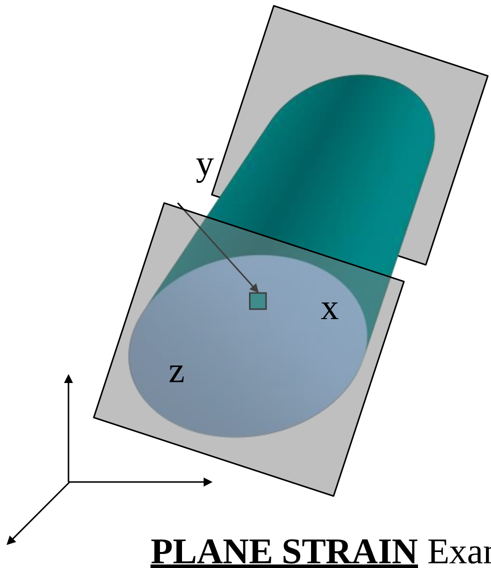
Nonzero **<u>strain</u>** components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

ε

Area

element dA



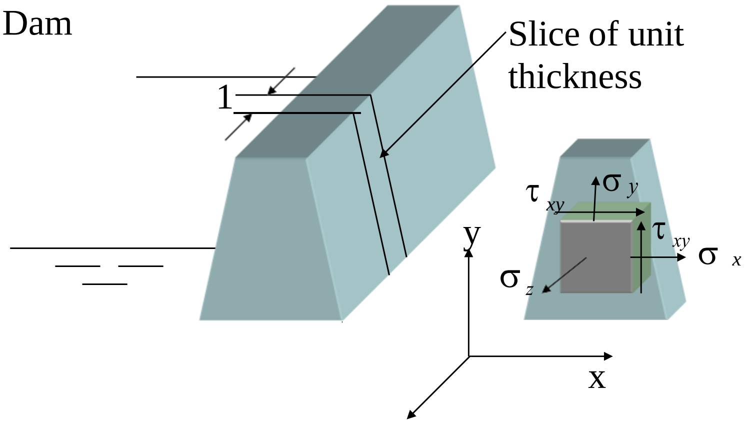


Assumptions:

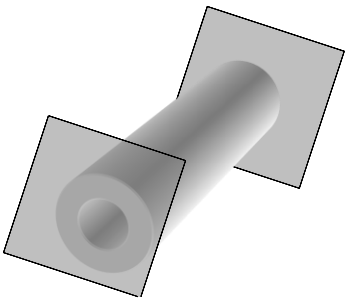
1. Displacement components u,v functions of (x,y) only and w=0
2. Top and bottom surfaces are fixed
3. X_c=0
4. p_x and p_y do not vary with z

<u>PLANE STRAIN</u> Examples:

1. Dam



2. Long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends



PLANE STRAIN

Nonzero **stress**: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$

Nonzero **strain** components: $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

Hence, the D matrix for the **plane strain case** is

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Principle of Minimum Potential Energy

Definition: For a linear elastic body subjected to body forces $\underline{X}=[X_a,X_b,X_c]^T$ and surface tractions $\underline{T}_S=[p_x,p_y,p_z]^T$, causing displacements $u=[u,v,w]^T$ and strains $\underline{\varepsilon}$ and stresses $\underline{\sigma}$, the **potential energy** Π is defined as the strain energy minus the potential energy of the loads involving $\underline{X}$ and $\underline{T}_S$

$$\Pi = U - W$$

Strain energy of the elastic body

Using the stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{D}\,\underline{\varepsilon}\, dV$$

<u>In 1D</u>

$$U = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_V E\varepsilon^2\, dV = \frac{1}{2}\int_{x=0}^{L} E\varepsilon^2\, A dx$$

<u>In 2D **plane stress** and **plane strain**</u>

$$U = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right) dV$$

Why?

<u>Principle of minimum potential energy:</u> Among all <u>admissible</u> displacement fields the one that satisfies the equilibrium equations also render the potential energy P a minimum.“admissible displacement field”:

1. **first derivative of the displacement components exist**
2. **satisfies the boundary conditions on S_u**

General procedure of Finite Element Method

Step 1: Divide structure into discrete elements (discretization).

- Divide the structure or solution region into subdivisions or elements. Hence, the structure is to be modeled with suitable finite elements.
- The number, type, size, and arrangement of the elements are to be decided.

Step 2: Select a proper interpolation or displacement model.

- Since the displacement solution of a complex structure under any specified loadconditions cannot be predicted exactly, we assume some suitable solution within anelement to approximate the unknown solution. The assumed solution must besimple from a computational standpoint, but it should satisfy certain convergencerequirements.
- In general, the solution or the interpolation model is taken in the formof a polynomial.

Step 3: Derive element stiffness matrices and load vectors.

- From the assumed displacement model finding,

Stiffness matrix – [K]e

Load vector. – [P]e

Step 4: Assemble element equations to obtain the overall equilibrium equations.

- Since the structure is composed of several finite elements, the individual element stiffness matrices and load vectors are to be assembled in a suitable manner and the overall equilibrium equations have to be formulated as

$$[K]\ [\phi] = [P]$$

Where [K] = the assembled stiffness matrix[ϕ]

= the vector of nodal displacements

[P] = Load vector

Step 5: Solve for the unknown nodal displacements.

The overall equilibrium equations have to be modified to account for the boundaryconditions of the problem. After the incorporation of the boundary conditions, theequilibrium equations can be expressed as

$$[K]\ [\phi] = [P]$$

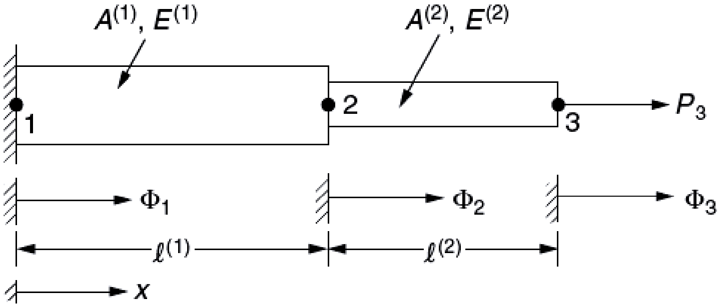
Step 6: Compute element strains and stresses.

From the known nodal displacementsϕ, if required, the element strains and stressescan be computed by using the necessary equations of solid or structural mechanics.

Example 1.1: A_1 = 200 mm^2, $E_1 = E_2 = E = 2 \times 10^6$ N/mm^2

A_2 = 100 mm^2, $l_1 = l_2$ = 100 mm

P_3 = 1000 N. Find: Displacement and stress & strain.



$$[K^{(1)}] = \frac{A^{(1)}E^{(1)}}{l^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} \Phi_1 & \Phi_2 \end{array} \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ \Phi_1 \\ \Phi_2 \end{array} \tag{E.16}$$

$$[K^{(2)}] = \frac{A^{(2)}E^{(2)}}{l^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} \Phi_2 & \Phi_3 \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ \Phi_2 \\ \Phi_3 \end{array} \tag{E.17}$$

Let overall stiffness matrix [K] = [K$^{(1)}$] + [K$^{(2)}$]

$$[\underset{\sim}{K}] = 10^6 \begin{array}{c} \begin{array}{ccc} \Phi_1 & \Phi_2 & \Phi_3 \end{array} \\ \begin{bmatrix} 4 & -4 & 0 \\ -4 & 4+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{array} = 2 \times 10^6 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

In the present case, external loads act only at the node points; as such, there is no need to assemblethe element load vectors. The overall or global load vector can be written as

$$\vec{P} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}$$

$$2 \times 10^6 = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}$$

$$2 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

By solving the matrix

$\Phi_2 = 0.25 \times 10^{-6}$cm and

$\Phi_3 = 0.75 \times 10^{-6}$cm

Derive element strains and stresses.

Once the displacements are computed, the strains in the elements can be found as

$$\varepsilon^{(1)} = \frac{\partial\phi}{\partial x} \text{ for element } 1 = \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} \equiv \frac{\Phi_2 - \Phi_1}{l^{(1)}} = 0.25 \times 10^{-7}$$

$$\varepsilon^{(2)} = \frac{\partial\phi}{\partial x} \text{ for element } 2 = \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}} \equiv \frac{\Phi_3 - \Phi_2}{l^{(2)}} = 0.50 \times 10^{-7}$$

The stresses in the elements are given by

$$\sigma^{(1)} = E^{(1)}\varepsilon^{(1)} = (2\times10^{7})(0.25\times10^{-7}) = 0.5 \text{ N/cm}^2$$

$$\sigma^{(2)} = E^{(2)}\varepsilon^{(2)} = (2\times10^{7})(0.50\times10^{-7}) = 1.0 \text{ N/cm}^2$$

Example 1.2:A thin plate as shown in Fig. (a) has uniform thickness of 2 cm and its modulus of elasticity is 200 x 10^3 N/mm^2 and density 7800 kg/m^3. In addition to its self-weight the plate is subjected to a point load P of500 N is applied at its midpoint.

Solve the following:

(i) Finite element model with two finite elements.

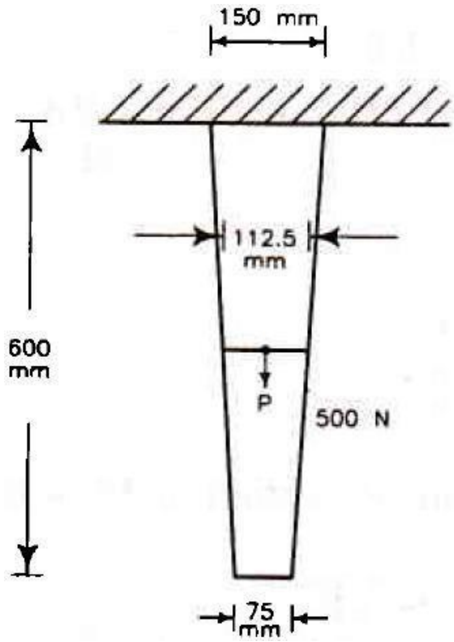
(ii) Global stiffness matrix.

(iii) Global load matrix.

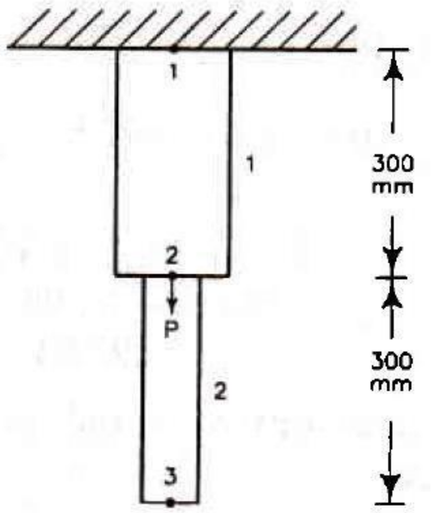
(iv) Displacement at nodal point.

(v) Stresses in each element.

(vi) Reaction at support.



(a)



(b)

(i) The tapered plate can be idealized as two element model with the tapered area converted to the rectangular equivalent area Refer Fig. (b). The areas A_1 and A_2

are equivalent areas calculated as

$$A_1 = \frac{15 + 11.25}{2} \times 2 = 26.25\ cm^2$$

$$A_2 = \frac{11.25 + 7.5}{2} \times 2 = 18.75\ cm^2$$

(ii) Global stiffness matrix can be obtained as

$$[k] = \frac{EA_1}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{EA_2}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{200 \times 10^3 \times 26.25 \times 10^2}{300}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{200 \times 10^3 \times 18.75 \times 10^2}{300}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= 0.175 \times 10^7 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0.125 \times 10^7 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= 10^7 \begin{bmatrix} 0.175 & -0.175 & 0 \\ -0.175 & 0.3 & -0.125 \\ 0 & -0.125 & 0.125 \end{bmatrix}$$

(ii) The load matrix given by

$$F = \rho \begin{bmatrix} \dfrac{A_1 L_1}{2} \\ \dfrac{A_1 L_1}{2} + \dfrac{A_2 L_2}{2} \\ \dfrac{A_2 L_2}{2} \end{bmatrix} + \begin{bmatrix} -R_1 \\ P \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{26.25 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} - R_1 \\ \dfrac{26.25 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} + \dfrac{18.75 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} + P \\ \dfrac{18.75 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 30.75 - R_1 \\ 30.75 + 21.93 + 500 \\ 21.93 \end{bmatrix}$$

(iv) The displacement at nodal point can be obtained by writing the equation in global form as

$$[k]\ [\delta] = [F]$$

$$10^7 \begin{bmatrix} 0.175 & -0.175 & 0 \\ -0.175 & 0.3 & -0.125 \\ 0 & -0.125 & 0.125 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 30.75 - R_1 \\ 552.68 \\ 21.93 \end{bmatrix}.$$

Using elimination approach and eleminating first row and column in which reaction occurs.

$$10^7 \begin{bmatrix} 0.3 & -0.125 \\ -0.125 & 0.125 \end{bmatrix} \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 552.68 \\ 21.93 \end{bmatrix}$$

$$\delta_1 = 0, \quad \delta_2 = 3.28 \times 10^{-4} \text{ mm}, \ \delta_3 = 3.45 \times 10^{-4} \text{ mm}.$$

(v) The stress in the element 1

$$6_1 = \frac{E}{L_1} [-1,\ 1] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \frac{200 \times 10^3}{300} \times 3.28 \times 10^{-4}$$

$$= 2.18 \times 10^{-1} \text{ MPa}$$

stress in the element 2

$$6_2 = \frac{E}{L_2} [-1,\ 1] \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \frac{200 \times 10^3}{300} [-\delta_2 + \delta_3] = 0.11 \times 10^{-1} \text{ MPa}$$

(vi) The reaction node 1

$$R_1 = \frac{E A_1}{L_1} [\delta_2 - 30.75]$$

$$= 0.175 \times 10^7 \times 3.28 \times 10^{-4} - 30.75 = 543.25.$$

Penalty Approach :

- In the preceding problems, the elimination approach was used to achieve simplified matrices. This method though simple, is not very easy to adapt in terms of algorithms written fix computer programs.
- An alternate method to achieve solutions is by the penalty approach. By this approach a rigid support is considered as a spring having infinite stiffness. Consider a system as shown in Fig.

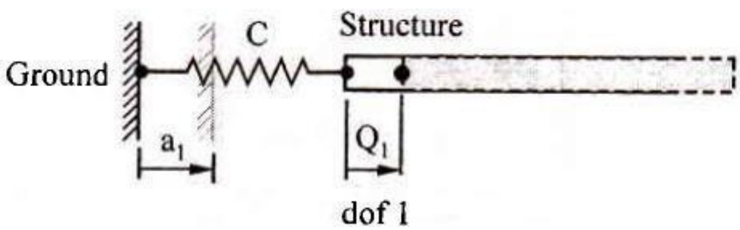


Fig. Penalty Approach

- The support or the ground is modelled with a high stiffness spring, having a stiffness C. To represent a rigid ground, c must be infinity.
- However, instead of introducing an infinite value in the calculations, a substantially high value of stiffness constant is introduced for those nodes resting on rigid supports.

– The magnitude of the stiffness constant should be at least 10^4 times more than the maximum value in the global stiffness matrix.

– From Fig. 1.6, it is seen that one end of the spring will displace by a_1. The displacement Q_1 (for dof l) will be approximately equal to a_1 as the spring has a high stiffness.

– Consider a simple 1D element with node 1 fixed.

$$\mathbf{KQ} = \mathbf{F}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

– At node l, the stiffness term is „C" is introduced to reflect the boundary condition related to a rigid support. To compensate this change, the force term will also be modified as:

$$\begin{bmatrix} k_{11}+C & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} F_1+Ca_1 \\ F_2 \end{Bmatrix}$$

– The reaction force as per penalty approach would be found by multiplying the added stiffness with the net deflection of the node.

$$R = - C(Q-a)$$

– The penalty approach is an approximate method and the accuracy of the forces depends on the value of C.

Example 1.3: Consider the bar shown in Fig.. An axial load $P = 200 \times 10^3$ N is applied as shown.Using the penalty approach for handling boundary conditions, do the following:

(a) Determine the nodal displacements

(b) Determine the stress in each material.

(c) Determine the reaction forces.

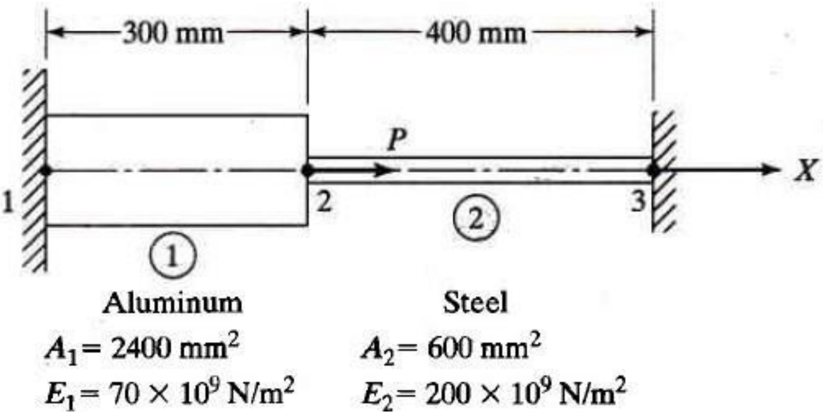


Fig. 1.7

(a) The element stiffness matrices are

$$\begin{array}{ccc} & 1 & 2 \leftarrow \text{Global dof} \\ \mathbf{k}^1 = \dfrac{70 \times 10^3 \times 2400}{300} & \left[\begin{array}{c} 1 \\ -1 \end{array}\right. & \left.\begin{array}{c} -1 \\ 1 \end{array}\right] \end{array}$$

and

$$\begin{array}{ccc} & 2 & 3 \\ \mathbf{k}^2 = \dfrac{200 \times 10^3 \times 600}{400} & \left[\begin{array}{c} 1 \\ -1 \end{array}\right. & \left.\begin{array}{c} -1 \\ 1 \end{array}\right] \end{array}$$

The structural stiffness matrix that is assembled from k^1 and k^2 is

$$\begin{array}{cccc} & 1 & 2 & 3 \\ \mathbf{K} = 10^6 & \left[\begin{array}{c} 0.56 \\ -0.56 \\ 0 \end{array}\right. & \begin{array}{c} -0.56 \\ 0.86 \\ -0.30 \end{array} & \left.\begin{array}{c} 0 \\ -0.30 \\ 0.30 \end{array}\right] \end{array}$$

The global load vector is

$$F = [0, 200 \text{ x } 10^3, 0]^T$$

Now dofs 1 and 3 are fixed. When using the penalty approach, therefore, a large numberC is added to thefirst and third diagonal elements of K. Choosing C

$$C = [0.86 \times 10^6] \times 10^4$$

Thus, the modified stiffness matrix is

$$\mathbf{K} = 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}$$

The finite element equations are given by

$$10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{Bmatrix}$$

which yields the solution

$$Q = [15.1432 \times 10^{-6}, 0.23257, 8.1127 \times 10^{-6}]\text{mm}$$

(b) The element stresses are

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{300} [-1 \quad 1] \begin{Bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{Bmatrix}$$
$$= 54.27 \text{ MPa}$$

where 1 MPa = 10^6 N/m^2= 1N/mm^2. Also,

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{400} [-1 \quad 1] \begin{Bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{Bmatrix}$$
$$= -116.29 \text{ MPa}$$

(c) The reaction forces are

$R_1 = -CQ_1$

$= -[0.86 \times 10^{10}] \times 15.1432 \times 10^6$

$= -130.23 \times 10^3$ N

$R_3 = -CQ_3$

$= -[0.86 \times 10^{10}] \times 8.1127 \times 10^6$

$= -69.77 \times 10^3$ N

Example 1.4:In Fig. (a), a load P = 60 x 10^3 N is applied as shown. Determine the displacement field,stress and support reactions in the body. Take E = 20 x 10^3N/mm^2.

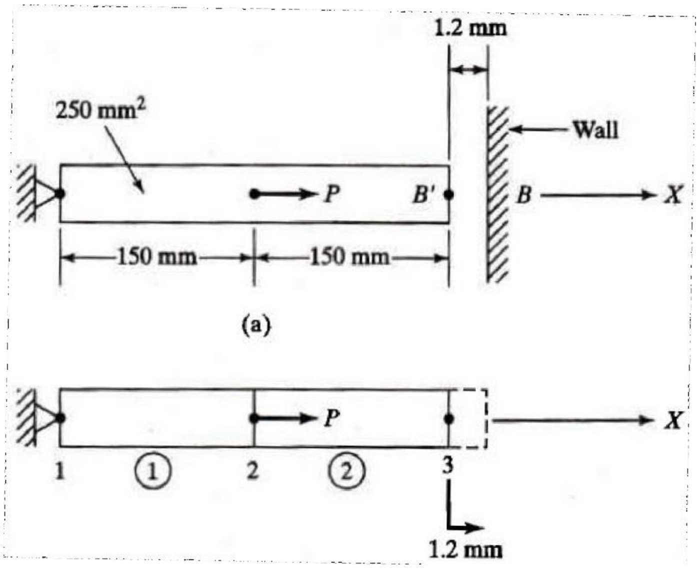


Fig.

The boundary conditions are $Q_1 = 0$ and $Q_3 = 1.2$ mm. The structural stiffness matrix K is

$$\mathbf{K} = \frac{20 \times 10^3 \times 250}{150}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

andthe global load vector F is

$$F = [0, 60 \times 10^3, 0]^T$$

In the penalty approach, the boundary conditions $Q_1 = 0$ and $Q_3 = 1.2$imply the followingmodifications: A large number C chosen here as $C = (2/3) \times 10^{10}$, is added on to the 1st and 3rd diagonal elements of K. Also, the number (C x 1.2) gets added on to the 3rd component of F. Thus, the modified equations are

The solution is

$$\frac{10^5}{3}\begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60.0 \times 10^3 \\ 80.0 \times 10^7 \end{Bmatrix}$$

$$Q = [7.49985 \times 10^{-5}, 1.500045, 1.200015]^T \text{mm}$$

The element stresses are

$$\sigma_1 = 200 \times 10^3 \times \frac{1}{150}[-1 \quad 1]\begin{Bmatrix} 7.49985 \times 10^{-5} \\ 1.500045 \end{Bmatrix}$$

$$= 199.996 \text{ MPa}$$

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{150}[-1 \quad 1]\begin{Bmatrix} 1.500045 \\ 1.200015 \end{Bmatrix}$$

$$= -40.004 \text{ MPa}$$

The reaction forces are

$$R_1 = -C \times 7.49985 \times 10^{-5}$$

$$= -49.999 \times 10^3 \text{N}$$

$$R_3 = -C \times (1.200015 - 1.2)$$

$= -10.001 \times 10^3 N$

Effect of Temperature on Elements:

When any material is subjected to a thermal stress, the thermal load is additional load acting on every element. This load can be calculated by using thermal expansion of the material due to the rise in. temperature.

Thermal stress in material can be given by

$\zeta_t = E\varepsilon_t$

Where ε_t= thermal strain

E = modulus of elasticity

$\varepsilon_t = \alpha \Delta t$

α= coefficient of linear expansion of material

Δt= change in temperature of material.

Then the thermal load is given by

Where, $Ft = \zeta_t A = AE\alpha \Delta t$

A = Area of the bar.

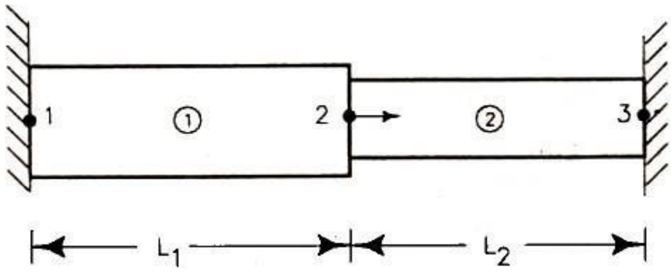


Fig.

Consider the horizontal step bar supported at two ends is subjected to athermal stress and load P at node 2 as shown in Fig. .

Thermal load in element 1

$$[F_1] = \begin{bmatrix} Ft_1 \\ Ft_{12} \\ 0 \end{bmatrix} = A_1 \; E \; \alpha \; \Delta t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Thermal load in element 2

$$[F_2] = \begin{bmatrix} 0 \\ Ft_{21} \\ Ft_3 \end{bmatrix} = A_2 \; E \; \alpha \; \Delta t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[F] = [F_1] + [F_2] + \begin{bmatrix} 0 \\ P \\ 0 \end{bmatrix} = \begin{bmatrix} -A_1 E\alpha\Delta t \\ A_1 E\alpha\Delta t - A_2 E\alpha\Delta t + P \\ A_2 E\alpha\Delta t \end{bmatrix}$$

Example 1.5 : An axial load $P = 300 \times 10^3$ N is applied at 20°C to the rod as shown in Fig. .The temperature is then raised to 60°C.

(a) Assemble the K and F matrices.

(b) Determine the nodal displacements and element stresses.

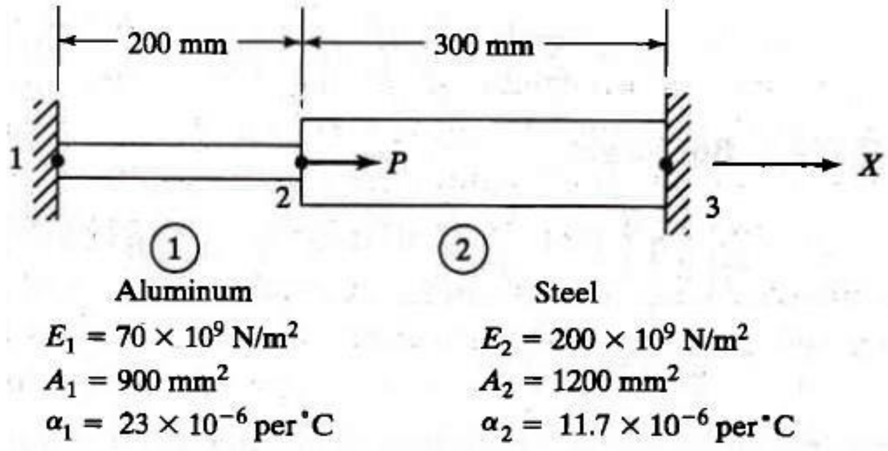


Fig.

(a) The element stiffness matrices are

$$\mathbf{k}^1 = \frac{70 \times 10^3 \times 900}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

$$\mathbf{k}^2 = \frac{200 \times 10^3 \times 1200}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

$$\mathbf{K} = 10^3\begin{bmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix} \text{N/mm}$$

Now, in assembling F, both temperature and point load effects have to be considered.

The element temperature forces due to ΔT = 40°C are obtained as

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \text{Global dof} \\ \boldsymbol{\Theta}^1 = 70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \quad \text{N} \end{array}$$

$$\boldsymbol{\Theta}^2 = 200 \times 10^3 \times 1200 \times 11.7 \times 10^{-6} \times 40 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \quad \text{N}$$

Upon assembling $\boldsymbol{\Theta}^1$, $\boldsymbol{\Theta}^2$, and the point load, we get

$$\mathbf{F} = 10^3\begin{Bmatrix} -57.96 \\ 57.96 - 112.32 + 300 \\ 112.32 \end{Bmatrix}$$

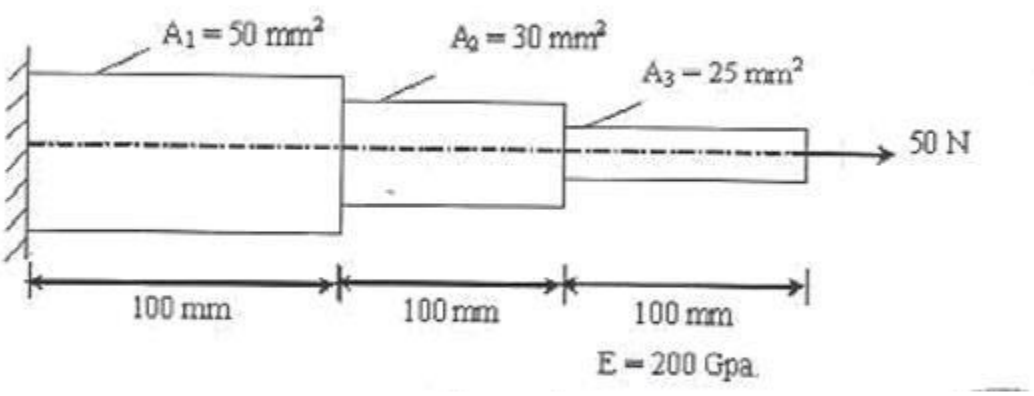
$$F = 10^3[-57.96, 245.64, 112.32]^T \text{N}$$

(b) The elimination approach will now be used to solve for the displacements. Since dofs 1 and 3 are fixed, the first and third rows and columns of K, together with the first and third components of F, are deleted. This results in the scalar equation

UNIT I
TUTORIAL QUESTIONS

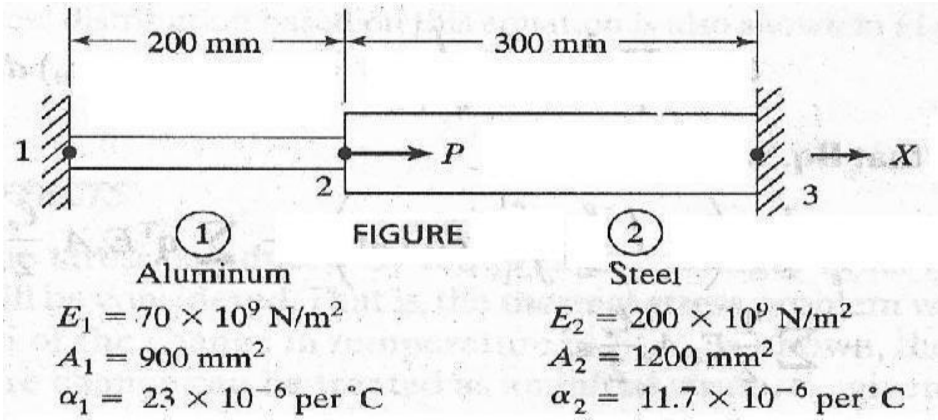
1. Derive the equations of equilibrium for 3D body
2. Explain about plane stress and plane strain
3. Describe advantages, disadvantages and applications of finite element analysis
4. The following equation is available for a physical phenomena
   $\frac{d^2y}{dx^2}$ -10 x^2 =5; 0<x<1, Boundary Conditions; y(0) =0, y(1) =0, Using Galarkin method of weighted residual find an approximate solution of the above differentialequation

5. Use Finite Element method Calculate nodal displacements and element stresses



UNIT I
ASSIGNMENT QUESTIONS

1. Describe the standard procedure to be followed for understanding the finiteelement method step by step with suitable example.
2. Derive the stiffness matrix of axial bar element with quadratic shape functions based on firstprinciples.
3. An axial load P=300KN is applied at 20^0 C to the rod as shown in Figure below.The temperature is the raised to 60^0 C .
   a) Assemble the K and F matrices.
   b) Determine the nodal displacements and stresses.



UNIT 2

TRUSSES
&
BEAMS

Syllabus

TRUSSES: Element matrices, assembling of global stiffness matrix, solution fordisplacements, reaction, stresses.
BEAMS: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses.

OBJECTIVE:

To learn the application of FEM equations for trusses and Beams

OUTCOME:

Derive element matrices to find stresses in trusses and Beams

UNIT II

Analysis of Trusses

- The links of a truss are two-force members, where the direction of loading is along the axis of the member. Every truss element is in direct tension or compression.
- All loads and reactions are applied only at the joints and all members are connected together at their ends by frictionless pin joints. This makes the truss members very similar to a 1D spar element.

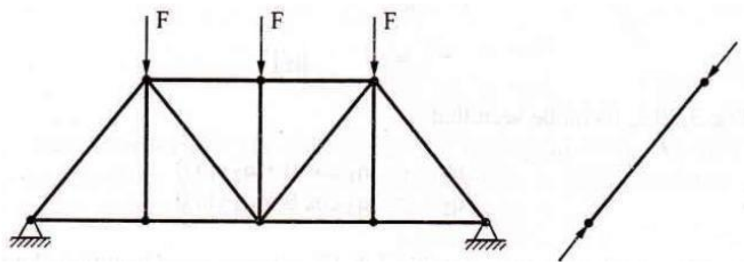
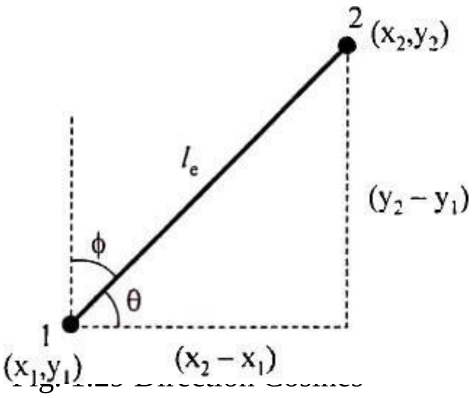


Fig. 1.28 Truss



The direction cosines l and$m$ can be expressed as:

$$l = \cos\theta = \frac{x_2 - x_1}{l_e}$$

$$m = \cos\phi = \sin\theta = \frac{y_2 - y_1}{l_e}$$

$$\sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

$q_1' = q_1 l + q_2\, m$

$q_2' = q_3\, l + q_4\, m$

$$[K] = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q$$

- **Thermal Effect In Truss Member**

(1) Thermal Load , P = $A_e E_e \varepsilon_e \begin{bmatrix} -l \\ -m \\ l \\ m \end{bmatrix}$

(2) Stress for an element, $\sigma = \dfrac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q - E_e\, \alpha\, \Delta t$

(3) Remaining steps will be same as earlier.

Example 1.6: A two member truss is as shown in Fig. The cross-sectional area ofeach member is 200 mm^2 and the modulus of elasticity is 200 GPa. Determine the deflections,reactions and stresses in each of the members.

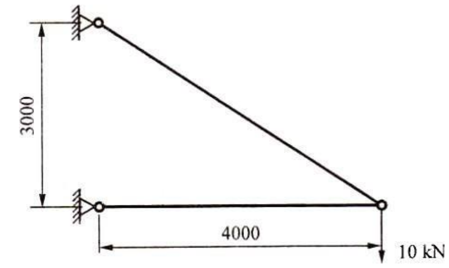


Fig.

In globalterms, each node would have 2 dof. These dof are marked as shown in Fig.1.31. The position of the nodes, with respect to origin (considered at node l) are as tabulated below:

Node	Xi	Yi
1	0	0
2	4000	0
3	0	3000

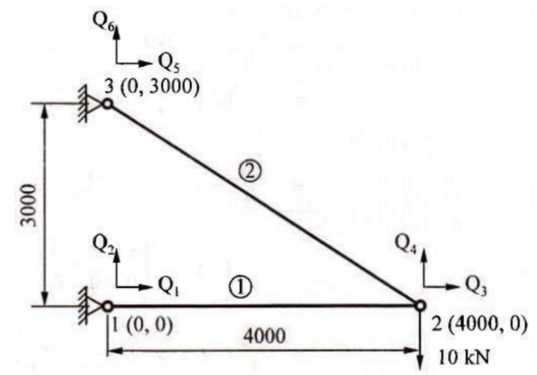


Fig.

For all elements, A=200 mm^2

and E= 200 x 10^3 N/mm^2

The element connectivity table with the relevant terms are:

Element	N_i	N_j	$l_e = \sqrt{(x_j - x_i)^2 - (y_j - y_i)^2}$	$\frac{A_e E_e}{l_e}$	$l = \frac{x_j - x_i}{l_e}$	$m = \frac{y_j - y_i}{l_e}$	l^2	m^2	lm
(1)	1	2	$\sqrt{(4000-0)^2-(0-0)^2}$ $= 4000$	10000	$\frac{4000-0}{4000}$ $= 1$	$\frac{0-0}{4000} = 0$	1	0	0
(2)	2	3	$\sqrt{(0-4000)^2+(3000-0)^2}$ $= 5000$	8000	$-\frac{4000}{5000}$ $= -0.8$	$\frac{3000}{5000} = 0.6$	0.64	0.36	-0.48

As each node has two dof in global form, for every element, the element stiffness matrix would be in a 4 x 4 form. For element 1 defined by nodes l-2,the dof are Q_1, Q_2, Q_3 andQ_4 and that for element 2 defined by nodes 2-3, would be Q_3, Q_4, Q_5 andQ_6.

Element 1: The element stiffness matrix would be :

$$
K^1 = 10 \times 10^3 \begin{array}{c} \overbrace{\begin{matrix} 1 & 2 \end{matrix}}^{\text{Node1}} \quad \overbrace{\begin{matrix} 3 & 4 \end{matrix}}^{\text{Node2}} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \Leftarrow \text{Global dof} \\ \begin{matrix} 1 \\ 2 \\ 3 \;\Downarrow \\ 4 \end{matrix} \end{array}
$$

$$
= 10^3 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \Leftarrow \text{Global dof} \\ \begin{matrix} 1 \\ 2 \\ 3 \;\Downarrow \\ 4 \end{matrix} \end{array}
$$

Element 2: The element stiffness matrix would be :

$$
K^2 = 8 \times 10^3 \begin{array}{c} \overbrace{\begin{matrix} 3 & 4 \end{matrix}}^{\text{Node2}} \quad \overbrace{\begin{matrix} 5 & 6 \end{matrix}}^{\text{Node3}} \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \end{array} \begin{array}{c} \Leftarrow \text{Global dof} \\ \begin{matrix} 3 \\ 4 \\ 5 \;\Downarrow \\ 6 \end{matrix} \end{array}
$$

$$
= 10^3 \begin{array}{c} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 5.12 & -3.84 & -5.12 & 3.84 \\ -0.48 & 2.88 & 3.84 & -2.88 \\ -5.12 & 3.84 & 5.12 & -3.84 \\ 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \end{array} \begin{array}{c} \Leftarrow \text{Global dof} \\ \begin{matrix} 3 \\ 4 \\ 5 \;\Downarrow \\ 6 \end{matrix} \end{array}
$$

The global stiffness matrix would be :

$$
K = 10^3 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & (10+5.12) & (0-3.84) & -5.12 & 3.84 \\ 0 & 0 & (0-3.84) & (0+2.88) & 3.84 & -2.88 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \end{array} \begin{array}{c} \Leftarrow \text{Global dof} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \;\Downarrow \\ 5 \\ 6 \end{matrix} \end{array}
$$

$$
= 10^3 \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \Leftarrow \text{Global dof}
$$

$$
= 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 15.12 & -3.84 & -5.12 & 3.84 \\ 0 & 0 & -3.84 & 2.88 & 3.84 & -2.88 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \Downarrow
$$

In this case, node 1 and node 3 are completely fixed and hence,

$$Q_1=Q_2=Q_5 = Q_6 = 0$$

Hence, rows and columns 1,2,5 and 6 can be eliminated

Also the external nodal forces,

$F_1 = F_2 = F_3 = F_5 = F_6 = 0$

$F4 = -10 \times 10^3$ N

The global force vector would be, $$\mathbf{F} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10\times 10^3 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} \text{Global dof} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \Downarrow$$

In global form, after using the elimination approach

$$\mathbf{KQ} = \mathbf{F}$$

$$10^3 \begin{bmatrix} 15.12 & -3.84 \\ -3.84 & 2.88 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10\times 10^3 \end{Bmatrix}$$

$$10^3 (15.12\, Q_3 - 3.84\, Q_4) = 0$$

$$Q_3 = 0.254\, Q_4$$

$10^3 (-3.84\, Q_3 + 2.88\, Q_4) = -10 \times 10^3$

$-3.84\, Q_3 + 2.88\, Q_4 = -10$

$-3.84\,(0.254\, Q_4) + 2.88\, Q_4 = -10$

$Q_4 = -5.25$mm

$Q_3 = -1.334$mm

The reactions can be found by using the equation:

$R = KQ - F$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_5 \\ R_6 \end{Bmatrix} = 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1.334 \\ -5.25 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$R_1 = -10 \times 10^3 \times (-1.334) = 13340$ N

$R_2 = 0$ N

$R_5 = -5.12 \times 10^3 \times (-1.334) + 3.84 \times 10^3 \times (-5.25) = -13340$ N

$R_6 = 3.84 \times 10^3 \times (-1.334) - 2.88 \times 10^3 \times (-5.25) = 9997.44$ N

To determine stresses: $\sigma = \dfrac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q$

Element 1:

$$\sigma_1 = \frac{200 \times 10^3}{4000} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1.334 \\ -5.25 \end{Bmatrix}$$

$$= -66.7 \text{ N/mm}^2$$

Element 2:

$$\sigma_2 = \frac{200 \times 10^3}{5000} \begin{bmatrix} 0.8 & -0.6 & -0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} -1.334 \\ -5.25 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 83.312 \text{ N/mm}^2$$

Example :Consider the four-bar truss shown in Fig. It is given that $E = 29.5 \times 10^6$ psi and $A_e = 1 \text{in.}^2$ for all elements. Complete the following:

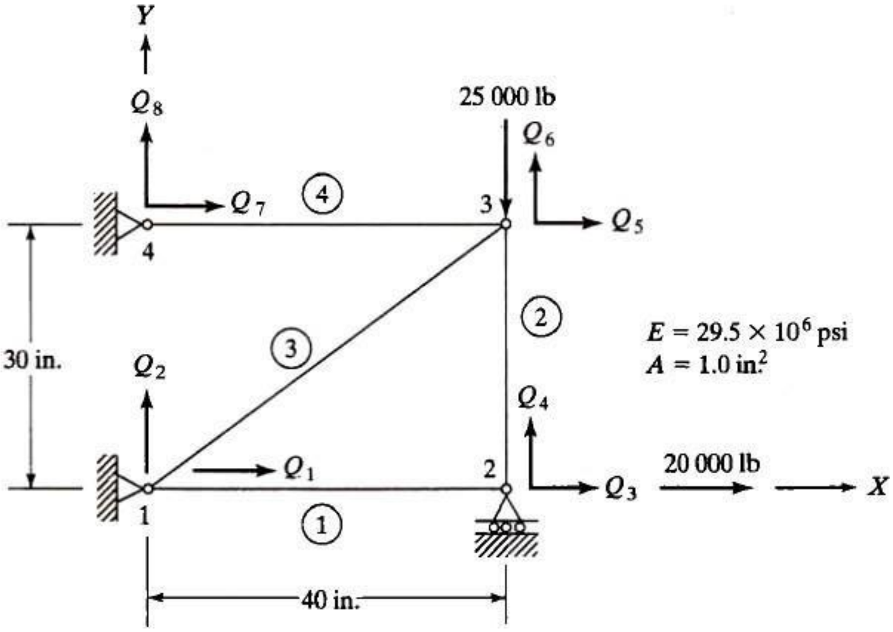
(a) Determine the element stiffness matrix for each element.

(b) Assemble the structural stiffness matrix K for the entire truss

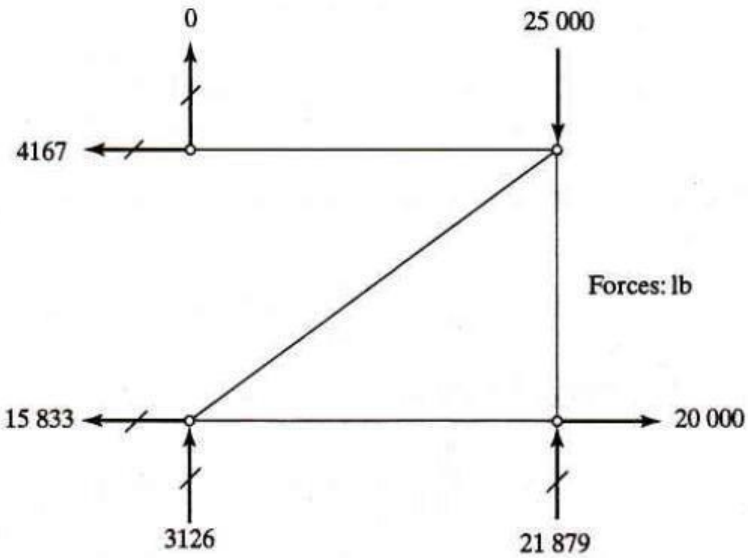
(c) Using the elimination approach, solve for the nodal displacement.

(d) Recover the stresses in each element.

(e) Calculate the reaction forces.



(a)



(b)

Fig

(a) It is recommended that a tabular form be used for representing nodal coordinate data and element information. The nodal coordinate data are as follows:

Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

The element connectivity table is

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

Note that the user has a choice in defining element connectivity. For example, the connectivityof element 2 can be defined as 2-3 instead of 3-2 as in the previous table. However,calculations of the direction cosines will be consistent with the adopted connectivityscheme. Using formulas,together with the nodal coordinate data andthe given element connectivity information, we obtain the direction cosines table:

Element	l_e	l	m
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0

For example, the direction cosines of elements 3 are obtained as

$l = (x_3 - x_1)l_e = (40 - 0)/50 = 0.8$ and $m = (y_3 - y_1)l_e = (30 - 0)/50 = 0.6$.

Now, the element stiffness matrices for element 1 can be written as

$$\mathbf{k}^1 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \leftarrow \text{Global dof} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

The global dofs associated with element 1, which is connected between nodes 1 and2,are indicated in k^1 earlier.These global dofs are shown in Fig. 1.32(a) and assistin assembling the various element stiffness matrices.The element stiffness matrices of elements 2,3and 4 are as follows:

$$\mathbf{k}^2 = \frac{29.5 \times 10^6}{30} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 3 \\ 4 \end{array}$$

$$\mathbf{k}^3 = \frac{29.5 \times 10^6}{50} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} .64 & .48 & -.64 & -.48 \\ .48 & .36 & -.48 & -.36 \\ -.64 & -.48 & .64 & .48 \\ -.48 & -.36 & .48 & .36 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$
\mathbf{k}^4 = \frac{29.5\times10^6}{40}\begin{array}{c} \begin{array}{cccc} 7 & 8 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 7 \\ 8 \\ 5 \\ 6 \end{array}
$$

(b) The structural stiffness matrix K is now assembled from the element stiffness matrices.By adding the element stiffness contributions, noting the element connectivity, we get

$$
\mathbf{K} = \frac{29.5\times10^6}{600}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\ -5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

(c) The structural stiffness matrix K given above needs to be modified to account for the boundary conditions. The elimination approach will be used here. The rows and columns corresponding to dofs 1, 2, 4, 7, and 8, which correspond to fixed supports, are deleted from the K matrix. The reduced finite element equations are given as

$$
\frac{29.5\times10^6}{600}\begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix}\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 20\,000 \\ 0 \\ -25\,000 \end{Bmatrix}
$$

Solution of these equations yields the displacements

$$
\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 27.12\times10^{-3} \\ 5.65\times10^{-3} \\ -22.25\times10^{-3} \end{Bmatrix}\text{ in.}
$$

The nodal displacement vector for the entire structure can therefore be written as

$Q = [0, 0, 27.12\text{x } 10^{-3}, 0, 5.65 \text{ x } 10^{-3}, -22.25 \text{ x } 10^{-3}, 0, 0]^T$ in.

(d) The stress in each element can now be determined as shown below.

The connectivity of element 1 is 1 - 2. Consequently, the nodaldisplacementvectorforelementlisgivenby $q = [0,0, 27.72 \text{ x } 10^{-3},0]^T$

$$
\sigma_1 = \frac{29.5\times10^6}{40}[-1 \quad 0 \quad 1 \quad 0]\begin{Bmatrix} 0 \\ 0 \\ 27.12\times10^{-3} \\ 0 \end{Bmatrix}
$$

$$
= 20\,000.0\text{ psi}
$$

The stress in member 2 is given by

$$\sigma_2 = \frac{29.5 \times 10^6}{30}\begin{bmatrix}0 & 1 & 0 & -1\end{bmatrix}\begin{Bmatrix} 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ -27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= -21\,880.0 \text{ psi}$$

Following similar steps, we get

ζ_3 = 5208.0Psi

ζ_4= 4L67.0Psi

(e) The final step is to determine the support reactions. We need to determine the reaction forces along dofs 1, 2, 4, 7and 8, which correspond to fixed supports. These are obtained by substituting for Q into the original finite element equation R = KQ - F. In this substitution, only those rows of K corresponding to the support dofs are needed, and F = 0 for those dofs. Thus, we have

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \frac{29.5 \times 10^6}{600}\begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

Which results in

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 3126.0 \\ 21879.0 \\ -4167.0 \\ 0 \end{Bmatrix} \text{lb}$$

UNIT II

BEAMS

Derivation of Shape Function for Beam Element [Fourth Order Beam Equation]

Consider the beam element as shown in Fig.. The beam is of length L with axial local co-ordinate x and transverse local co-ordinate y. The local transverse nodal displacements are given by d_{1y} and d_{2y}. The rotations are given by ϕ_1 and ϕ_2. The local nodal forces are given by F_{1y} and F_{2y}. The bending moments are given by m_1 and m_2.

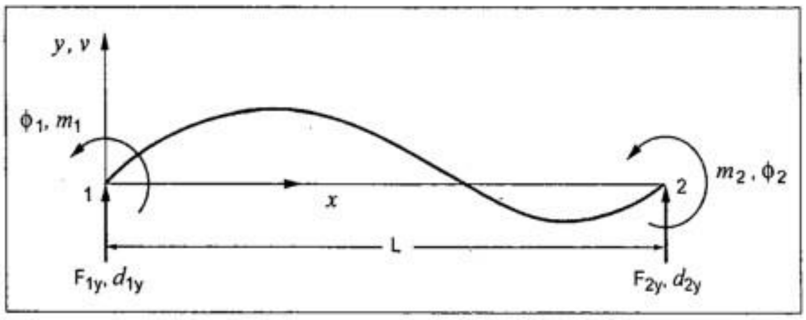


Fig. . Beam element with positive nodal displacements, rotations, forces, and moments

At all nodes, the following sign conversions are used.

(*i*) Moments are positive in the counterclockwise direction.

(*ii*) Rotations are positive in the counterclockwise direction.

(*iii*) Forces are positive in the positive y direction.

(*iv*) Displacements are positive in the positive y direction.

Fig. indicates the sign conventions used in simple beam theory for positive shear forces F and bending moments m.

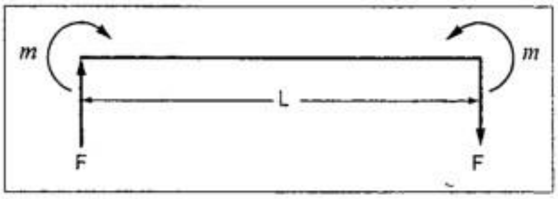


Fig. . Beam theory sign conventions for shear forces and bending moments

Assume the transverse displacement variation through the element length to be

$$v(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

We express v in terms of the nodal degrees of freedom d_{1y}, d_{2y}, ϕ_1 and ϕ_2 as follows:

At $x = 0$,

$$v(0) = a_4 = d_{1y}$$

$$\frac{dv_{(x)}}{dx} = 3 a_1 x^2 + 2 a_2 x + a_3$$

$$\frac{dv_{(0)}}{dx} = a_3 = \phi_1$$

When $x = L$,

$$v(L) = a_1 L^3 + a_2 L^2 + a_3 L + a_4 = d_{2y}$$

$$\frac{dv_{(L)}}{dx} = 3a_1 L^2 + 2a_2 L + a_3 = \phi_2$$

where $\phi = \frac{dv}{dx}$

Finding a_1 and a_2 in terms of d_{1y}, d_{2y}, ϕ_1 and ϕ_2 by using the above equations ...,

$$\Rightarrow \quad d_{2y} = a_1 L^3 + a_2 L^2 + a_3 L + a_4$$

$$= a_1 L^3 + a_2 L^2 + a_3 L + d_{1y} \qquad [\because a_4 = d_{1y}]$$

$$\Rightarrow \quad (d_{2y} - d_{1y}) = a_1 L^3 + a_2 L^2 + \phi_1 L$$

$$\Rightarrow \quad (d_{2y} - d_{1y} - \phi_1 L) = a_1 L^3 + a_2 L^2$$

$$\Rightarrow \quad \frac{1}{L}(d_{2y} - d_{1y} - \phi_1 L) = a_1 L^2 + a_2 L$$

$$\Rightarrow \quad \phi_2 = 3a_1 L^2 + 2a_2 L + a_3$$

$$= 3a_1 L^2 + 2a_2 L + \phi_1 \qquad [\because a_3 = \phi_1]$$

$$\Rightarrow \quad \phi_2 - \phi_1 = 3a_1 L^2 + 2a_2 L$$

Equation

$$\Rightarrow \quad \frac{3}{L}(d_{2y} - d_{1y} - \phi_1 L) = 3a_1 L^2 + 3a_2 L$$

Solving equation

$$\phi_2 - \phi_1 = 3a_1 L^2 + 2a_2 L$$

$$\frac{3}{L}(d_{2y} - d_{1y} - \phi_1 L) = 3a_1 L^2 + 3a_2 L$$

Subtracting,

$$\phi_2 - \phi_1 - \frac{3}{L}(d_{2y} - d_{1y} - \phi_1 L) = -a_2 L$$

$$\phi_2 - \phi_1 - \frac{3}{L}(d_{2y} - d_{1y}) + \frac{3}{L}\phi_1 L = -a_2 L$$

$$\phi_2 - \phi_1 - \frac{3}{L}(d_{2y} - d_{1y}) + 3\phi_1 = -a_2 L$$

$$\phi_2 + 2\phi_1 - \frac{3}{L}(d_{2y} - d_{1y}) = -a_2 L$$

$$\frac{1}{L}(\phi_2 + 2\phi_1) - \frac{3}{L^2}(d_{2y} - d_{1y}) = -a_2$$

$$\Rightarrow \quad \frac{-1}{L}(\phi_2 + 2\phi_1) + \frac{3}{L^2}(d_{2y} - d_{1y}) = a_2$$

$$\Rightarrow \quad \frac{-1}{L}(\phi_2 + 2\phi_1) - \frac{3}{L^2}(d_{1y} - d_{2y}) = a_2$$

$$\Rightarrow \quad a_2 = \frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)$$

Substitute a_2 value in equation

$$\Rightarrow \quad \phi_2 = 3a_1 L^2 + 2L\left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right] + a_3$$

$$= 3a_1 L^2 - \frac{6}{L}(d_{1y} - d_{2y}) - 2(2\phi_1 + \phi_2) + \phi_1 \qquad [\because a_3 = \phi_1]$$

$$\Rightarrow \quad \phi_2 - \phi_1 = 3a_1 L^2 - \frac{6}{L}(d_{1y} - d_{2y}) - 4\phi_1 - 2\phi_2$$

$$\Rightarrow \quad 3\phi_1 + 3\phi_2 = 3a_1 L^2 - \frac{6}{L}(d_{1y} - d_{2y})$$

$$\Rightarrow \quad 3a_1 L^2 = 3\phi_1 + 3\phi_2 + \frac{6}{L}(d_{1y} - d_{2y})$$

$$a_1 L^2 = \phi_1 + \phi_2 + \frac{2}{L}(d_{1y} - d_{2y})$$

$$\Rightarrow \quad a_1 = \frac{1}{L^2}(\phi_1 + \phi_2) + \frac{2}{L^3}(d_{1y} - d_{2y})$$

$$\Rightarrow \quad \boxed{a_1 = \frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)}$$

Substitute a_1, a_2, a_3 and a_4 values in equation

$$v(x) = \left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + d_{1y}$$

$$[\because a_3 = \phi_1;\ a_4 = d_{1y}]$$

In matrix form, $\quad v(x) = [N]\{d\}$

$$\Rightarrow \quad v(x) = [N_1 \ \ N_2 \ \ N_3 \ \ N_4]\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

$$\Rightarrow \quad v(x) = N_1 d_{1y} + N_2 \phi_1 + N_3 d_{2y} + N_4 \phi_2$$

where N_1, N_2, N_3 and N_4 are shape functions for beam element.

Stiffness Matrix [K] for Beam Element

The stiffness matrix for the beam element is derived by using a direct equilibrium approach and beam theory sign conversions.

We know that,

Transverse displacement

$$v(x) = \left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + d_{1y}$$

$$\Rightarrow \quad \frac{dv_{(x)}}{dx} = 3x^2\left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] + 2x\left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right] + \phi_1$$

$$+\ 2\left[\frac{-3}{L^2}(d_{1y}-d_{2y})-\frac{1}{L}(2\,\phi_1+\phi_2)\right]$$

$$\frac{d^3v(x)}{dx^3} = 6\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

Put $x = 0$ in equation (:

$$\Rightarrow \quad \frac{d^2v(0)}{dx^2} = 0 + 2\left[\frac{-3}{L^2}(d_{1y}-d_{2y})-\frac{1}{L}(2\,\phi_1+\phi_2)\right]$$

$$= \frac{-6}{L^2}(d_{1y}-d_{2y})-\frac{2}{L}(2\,\phi_1+\phi_2)$$

$$= \frac{1}{L^3}\left[-6\,L\,d_{1y}+6\,L\,d_{2y}-4\,L^2\,\phi_1-2\,L^2\,\phi_2\right]$$

$$\frac{d^2v(0)}{dx^2} = \frac{1}{L^3}\left[-6\,L\,d_{1y}-4\,L^2\,\phi_1+6\,L\,d_{2y}-2\,L^2\,\phi_2\right]$$

Put $x = L$ in equation

$$\Rightarrow \quad \frac{d^2v(L)}{dx^2} = 6\,L\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

$$+\ 2\left[\frac{-3}{L^2}(d_{1y}-d_{2y})-\frac{1}{L}(2\,\phi_1+\phi_2)\right]$$

$$= \frac{12\,L}{L^3}(d_{1y}-d_{2y})+\frac{6\,L}{L^2}(\phi_1+\phi_2)-\frac{6}{L^2}(d_{1y}-d_{2y})-\frac{2}{L}(2\,\phi_1+\phi_2)$$

$$= \frac{1}{L^3}\ [12\,L\,d_{1y}-12\,L\,d_{2y}+6\,L^2\,\phi_1+6\,L^2\,\phi_2-6\,L\,d_{1y}$$

$$+6\,L\,d_{2y}-4\,L^2\,\phi_1-2\,L^2\,\phi_2]$$

$$\frac{d^2v(L)}{dx^2} = \frac{1}{L^3}\ [6\,L\,d_{1y}+2\,L^2\,\phi_1-6\,L\,d_{2y}+4\,L^2\,\phi_2]$$

Put $x = 0$ in equation ,

$$\frac{d^3v(0)}{dx^3} = 6\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

$$= \frac{1}{L^3}\ [12\,d_{1y}-12\,d_{2y}+6\,L\,\phi_1+6\,L\,\phi_2]$$

$$\frac{d^3v(0)}{dx^3} = \frac{1}{L^3}\ [12\,d_{1y}+6\,L\,\phi_1-12\,d_{2y}+6\,L\,\phi_2]$$

Put $x = L$ in equation

$$\Rightarrow \quad \frac{d^3v(L)}{dx^3} = 6\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

$$= \frac{1}{L^3}\ [12\,d_{1y}-12\,d_{2y}+6\,L\,\phi_1+6\,L\,\phi_2]$$

$$\frac{d^3v(L)}{dx^3} = \frac{1}{L^3}\ [12\,d_{1y}+6\,L\,\phi_1-12\,d_{2y}+6\,L\,\phi_2]$$

We know that,

Nodal force, $F_{1y} = E\,I\ \dfrac{d^3v(0)}{dx^3}$

$$\Rightarrow \quad F_{1y} = \frac{EI}{L^3}\ [12\,d_{1y}+6\,L\,\phi_1-12\,d_{2y}+6\,L\,\phi_2]$$

Bending moment, $m_1 = -EI \frac{d^2v(0)}{dx^2}$

$$= \frac{-EI}{L^3} [-6\,L\,d_{1y} - 4\,L^2 \phi_1 + 6\,L\,d_{2y} - 2\,L^2 \phi_2]$$

$$m_1 = \frac{EI}{L^3} [6\,L\,d_{1y} + 4\,L^2 \phi_1 - 6\,L\,d_{2y} + 2\,L^2 \phi_2]$$

Nodal force, $F_{2y} = -EI \frac{d^3v(L)}{dx^3}$

$$= \frac{-EI}{L^3} [12\,d_{1y} + 6\,L\,\phi_1 - 12\,d_{2y} + 6\,L\,\phi_2]$$

$$F_{2y} = \frac{EI}{L^3} [-12\,d_{1y} - 6\,L\,\phi_1 + 12\,d_{2y} - 6\,L\,\phi_2]$$

Bending moment, $m_2 = EI \frac{d^2v(L)}{dx^2}$

$$\Rightarrow \quad m_2 = \frac{EI}{L^3} [6\,L\,d_{1y} + 2\,L^2 \phi_1 - 6\,L\,d_{2y} + 4\,L^2 \phi_2]$$

Arranging to the above equation (F_{1y}, m_1, F_{2y}, m_2) in matrix form,

$$\Rightarrow \quad \begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6\,L & -12 & 6\,L \\ 6\,L & 4\,L^2 & -6\,L & 2\,L^2 \\ -12 & -6\,L & 12 & -6\,L \\ 6\,L & 2\,L^2 & -6\,L & 4\,L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} \ldots$$

This is a finite element equation for a beam element.

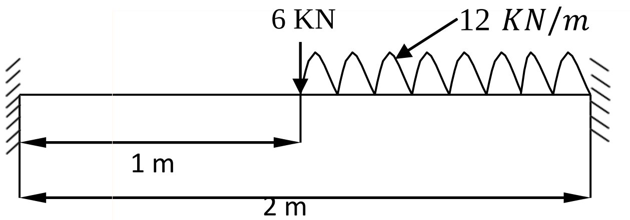
$$\text{Stiffness matrix, } [K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6\,L & -12 & 6\,L \\ 6\,L & 4\,L^2 & -6\,L & 2\,L^2 \\ -12 & -6\,L & 12 & -6\,L \\ 6\,L & 2\,L^2 & -6\,L & 4\,L^2 \end{bmatrix}$$

where E = Young's modulus

I = Moment of inertia

L = Length of the beam

For the beam and loading shown in fig. calculate the nodal displacements.
Take [E] =210 GPa =210×10⁹ N/m^2 , [I] = 6×10⁻⁶ m⁴ NOV / DEC 2013



Given data

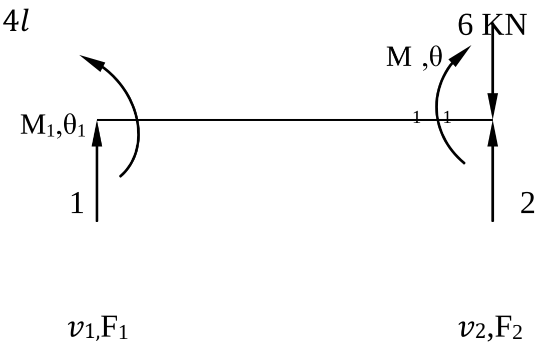
Young's modulus [E] =210 GPa =210×10^9 N/m^2
Moment of inertia [I] = 6×10^{-6} m^4
Length $[L]_1$ = 1m
Length $[L]_2$ = 1m
W=12 kN/m =12×10^3 N/m
F = 6KN

To find

- Deflection

Formula used

$$f(x)\begin{Bmatrix} \frac{-l}{2} \\ \frac{-l^2}{12} \\ \frac{-l}{2} \\ \frac{l^2}{12} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$



Solution

For element 1

$$f(x)\begin{Bmatrix} \frac{-l}{2} \\ \frac{-l^2}{12} \\ \frac{-l}{2} \\ \frac{l^2}{12} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

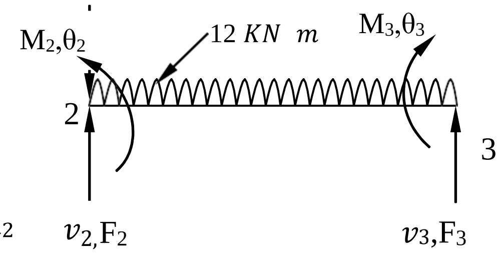
Applying boundary conditions

F_1=0N ; F_2=-6KN=-6×10^3 N; f(x)=0

$M_1=M_2=0$; $u_1=0$; $\theta_1=0$; $u_2\neq 0$; $\theta_2\neq 0$

$$10^3\times\begin{bmatrix}0\\0\\-6\\0\end{bmatrix}=\frac{210\times10^{9}\times6\times10^{-6}}{1^3}\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{bmatrix}u_1\\ \theta_1\\ u_2\\ \theta_2\end{bmatrix}$$

$$=1.26\times10^{6}\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_2\\ 0\end{Bmatrix}$$



For element 2

$$f(x)\begin{Bmatrix}\frac{-l}{2}\\ \frac{-l^2}{12}\\ \frac{-l}{2}\\ \frac{l^2}{12}\end{Bmatrix}+\begin{Bmatrix}F_2\\ M_2\\ F_3\\ M_3\end{Bmatrix}=\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ 12 & -6l & 12 & -6\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ u_3\\ \theta_3\end{Bmatrix}$$

Applying boundary conditions

$f(x) = -12\ kN/m = 12\times10^{3}\ N/m$; $F_2=F_3=0=M_2=M_3$

$u_2\neq0$; $\theta_2\neq0$; $u_3=\theta_3=0$

$$10^{3}\times\begin{Bmatrix}-6\\ -1\\ -6\\ 1\end{Bmatrix}+\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}=1.26\times10^{6}\times\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & 6 & 12 & -6\\ 6 & 4 & -6 & 4\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

$$10^{3}\times\begin{Bmatrix}-6\\ -1\\ -6\\ 1\end{Bmatrix}=1.26\times10^{6}\times\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & 6 & 12 & -6\\ 6 & 4 & -6 & 4\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

Assembling global matrix

$$10^{3}\times\begin{Bmatrix}0\\ 0\\ -12\\ -1\\ -6\\ 1\end{Bmatrix}=1.26\times10^{6}\times\begin{bmatrix}12 & 6 & -12 & -6 & 0 & 0\\ 6 & 4 & -6 & 2 & 0 & 0\\ -12 & -6 & 24 & 0 & -12 & 6\\ 6 & 2 & 0 & 8 & -6 & 2\\ 0 & 0 & -12 & -6 & 12 & -6\\ 0 & 0 & 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

Solving matrix

$-12\times10^{3}=1.26\times10^{6}\times24u_2=0$; $u_2=-3.96\times10^{-4}$m

$-1\times10^{3}=1.26\times10^{6}\times8\theta_2=0$; $\theta_2=-9.92$rad

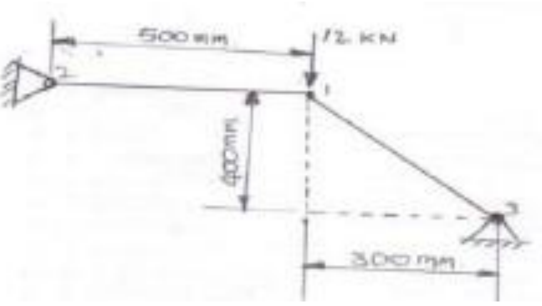
Result

$\theta_2=-9.92$rad

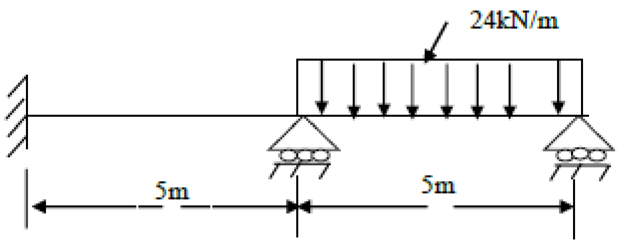
$u_2=-3.96\times10^{-4}$m

UNIT II
Tutorial Questions

1. For the two bar truss shown in figure, determine the displacement at node 1 and stresses in element2, Take E=70GPa, A= 200mm^2.



2. For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=4x10^{6}m^4.



3. For a beam and loading shown in fig., determine the slopes at 2 and 3 and the vertical deflection at the midpoint of the distributed load.

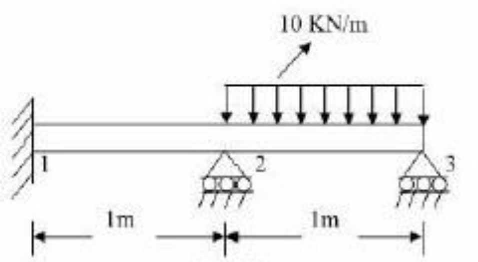
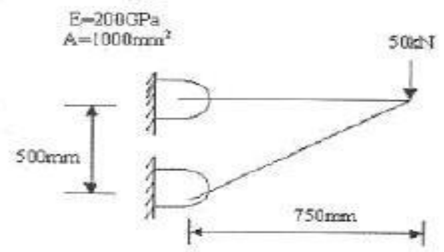


Fig.5

4.

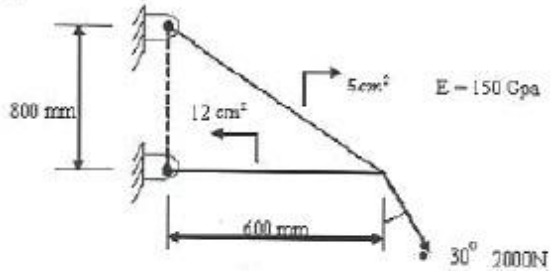
Determine the stiffness matrix, stresses and reactions in the truss structure shown in figure



UNIT II
Assignment Questions

1. Derive the stiffness matrix for Truss element
2. Derive the stiffness matrix for Beam element
3.

Calculate the nodal displacement, stresses and support reactions for the truss shown in Figure.



UNIT 3

TWO DIMENSIONL PROBLEMS

&

AXI-SYMMETRIC MODELS

Syllabus

Two Dimensional Problems: Basic concepts of plane stress and plane strain, stiffness matrix of CST element, finite element solution of plane stress problems. Axi-Symmetric Model: Finite element modelling of axi-symmetric solids subjected to axi- symmetric loading with triangular elements.

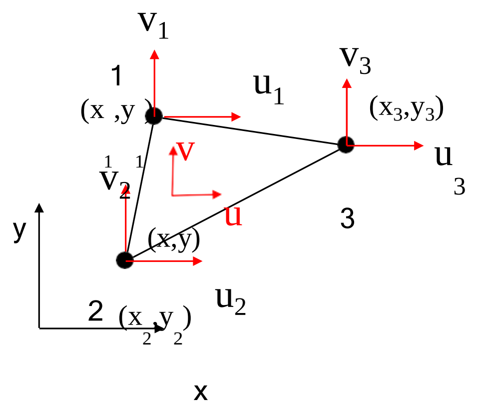
OBJECTIVE:

To learn the applications of FEM equations in 2D Plane problems with CST elements.

OUTCOME:

Formulate FE characteristic equations for axisymmetric problems and analyze plain stress, plain strain and Derive element matrices for CST elements.

<u>Constant Strain Triangle (CST) :</u> Simplest 2D finite element



- 3 nodes per element
- 2 dofs per node (each node can move in x- and y- directions)
- Hence 6 dofs per element

The <u>displacement approximation</u> in terms of shape functions is

$$u\ (x,y) \approx N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v(x,y) \approx N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\underline{u} = \begin{Bmatrix} u\ (x,y) \\ v\ (x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

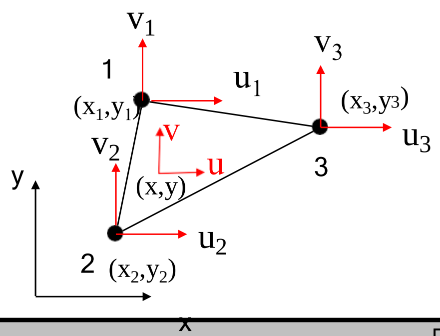
$$\boxed{\underline{u}_{2\times 1} = \underline{N}_{2\times 6}\ \underline{d}_{6\times 1}}$$

$$\underline{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

Formula for the shape functions are

$$N_1 = \frac{a_1 + b_1 x + c_1 y}{2A}$$
$$N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$$
$$N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$$

where

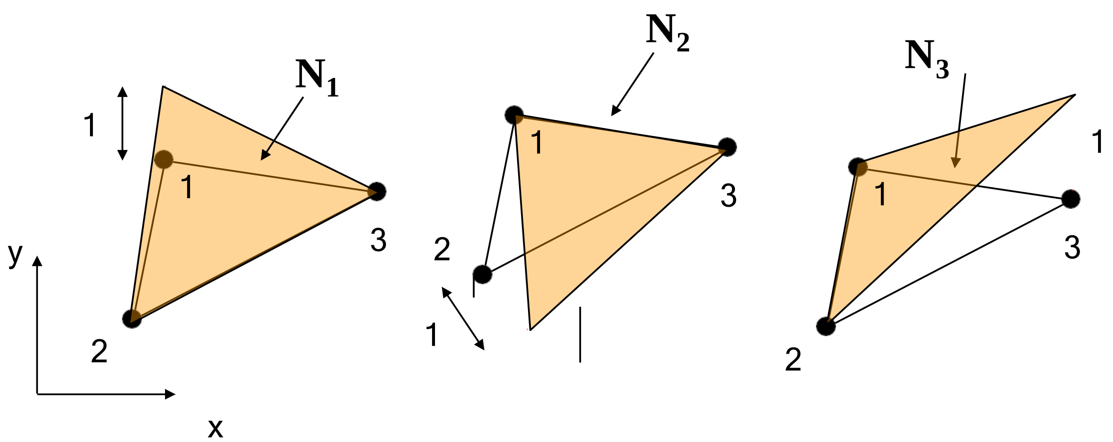


$$A = area\ of\ triangle = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2 & b_1 &= y_2 - y_3 & c_1 &= x_3 - x_2 \\ a_2 &= x_3 y_1 - x_1 y_3 & b_2 &= y_3 - y_1 & c_2 &= x_1 - x_3 \\ a_3 &= x_1 y_2 - x_2 y_1 & b_3 &= y_1 - y_2 & c_3 &= x_2 - x_1 \end{aligned}$$

Properties of the shape functions:

1. The shape functions N_1, N_2 and N_3 are linear functions of x and y



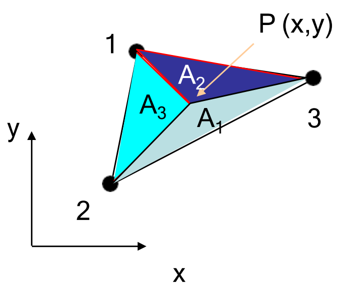
$$N_i = \begin{cases} 1 & at\ node\ 'i' \\ 0 & at\ other\ nodes \end{cases}$$

2. At every point in the domain

$$\boxed{\begin{aligned}\sum_{i=1}^{3} N_i &= 1\\ \sum_{i=1}^{3} N_i x_i &= x\\ \sum_{i=1}^{3} N_i y_i &= y\end{aligned}}$$

3. Geometric interpretation of the shape functions

At any point P(x,y) that the shape functions are evaluated,



$$\boxed{\begin{aligned}N_1 &= \frac{A_1}{A}\\ N_2 &= \frac{A_2}{A}\\ N_3 &= \frac{A_3}{A}\end{aligned}}$$

Approximation of the strains

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \approx \underline{B}\underline{d}$$

$$\underline{B} = \begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} & 0 & \frac{\partial N_2(x,y)}{\partial x} & 0 & \frac{\partial N_3(x,y)}{\partial x} & 0 \\ 0 & \frac{\partial N_1(x,y)}{\partial y} & 0 & \frac{\partial N_2(x,y)}{\partial y} & 0 & \frac{\partial N_3(x,y)}{\partial y} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial x} \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Inside each element, all components of strain are constant: hence the name **Constant Strain Triangle** Element stresses (constant inside each element)

$$\boxed{\underline{\sigma} = \underline{D}\,\underline{B}\,\underline{d}}$$

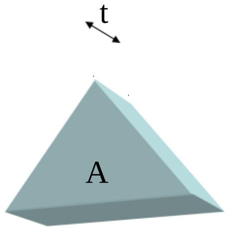
IMPORTANT NOTE:

1. The displacement field is continuous across element boundaries

2. The strains and stresses are NOT continuous across element boundaries

Element stiffness matrix

$$\boxed{\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\, \underline{B}\, dV}$$



Since $\underline{B}$ is constant

$$\underline{k} = \underline{B}^T \underline{D}\, \underline{B} \int_{V^e} dV = \underline{B}^T \underline{D}\, \underline{B}\, At$$

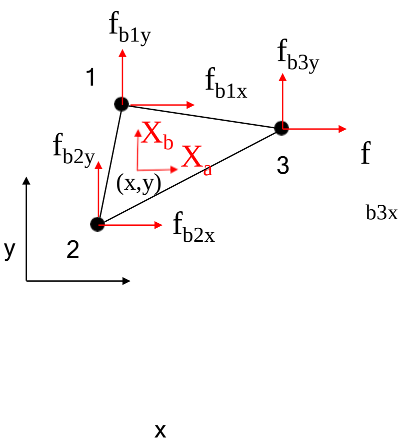
t=thickness of the element
A=surface area of the element

Element nodal load vector

$$\boxed{\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\, dS}_{\underline{f}_s}}$$

Element nodal load vector due to body forces

$$\boxed{\underline{f}_b = \int_{V^e} \underline{N}^T \underline{X}\, dV = t\int_{A^e} \underline{N}^T \underline{X}\, dA}$$



$$\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a\, dA \\ t\int_{A^e} N_1 X_b\, dA \\ t\int_{A^e} N_2 X_a\, dA \\ t\int_{A^e} N_2 X_b\, dA \\ t\int_{A^e} N_3 X_a\, dA \\ t\int_{A^e} N_3 X_b\, dA \end{Bmatrix}$$

EXAMPLE:

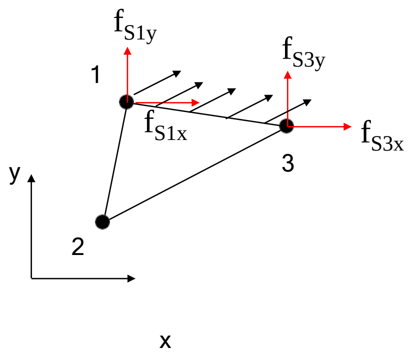
If $X_a=1$ and $X_b=0$

$$\underline{f}_b=\begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix}=\begin{Bmatrix} t\int_{A^e} N_1 X_a\, dA \\ t\int_{A^e} N_1 X_b\, dA \\ t\int_{A^e} N_2 X_a\, dA \\ t\int_{A^e} N_2 X_b\, dA \\ t\int_{A^e} N_3 X_a\, dA \\ t\int_{A^e} N_3 X_b\, dA \end{Bmatrix}=\begin{Bmatrix} t\int_{A^e} N_1\, dA \\ 0 \\ t\int_{A^e} N_2\, dA \\ 0 \\ t\int_{A^e} N_3 dA \\ 0 \end{Bmatrix}=\begin{Bmatrix} \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \end{Bmatrix}$$

Element nodal load vector due to traction

$$\boxed{\underline{f}_S=\int_{S_T^e} \underline{\mathrm{N}}^T \underline{T}_S\, dS}$$

EXAMPLE:



$$\underline{f}_S=t\int_{l_{1-3}^e} \underline{\mathrm{N}}^T\Big|_{along 1-3} \underline{T}_S\, dS$$

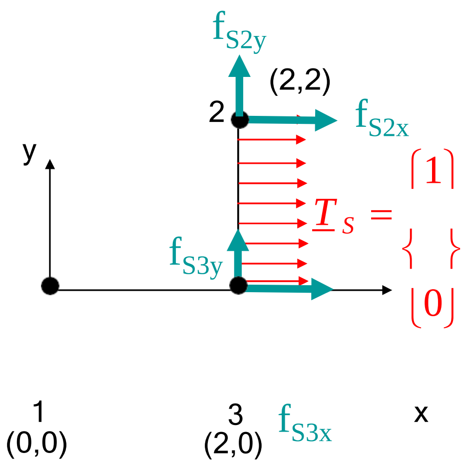
Recommendations for use of CST

1. Use in areas where strain gradients are small
2. Use in mesh transition areas (fine mesh to coarse mesh)

3. Avoid CST in critical areas of structures (e.g., stress concentrations, edges of holes, corners)
4. In general CSTs are not recommended for general analysis purposes as a very large number of these elements are required for reasonable accuracy.

Element nodal load vector due to traction

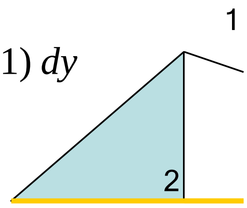
EXAMPLE:



$$\underline{f}_S = t\int_{l_{2-3}^e} \underline{\mathrm{N}}^T\Big|_{along\,2-3} \underline{T}_S \, dS$$

$$f_{S_{2x}} = t\int_{l_{2-3}^e} N_2\Big|_{along\,2-3} (1)\, dy$$

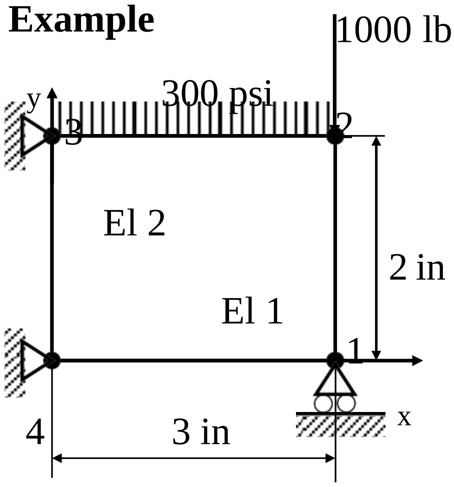
$$= t\left(\frac{1}{2}\right) \times 2 \times 1 = t$$



Similarly, compute

$$f_{S_{2y}} = 0$$
$$f_{S_{3x}} = t$$
$$f_{S_{3y}} = 0$$

Example



Thickness (t) = 0.5 in
E= 30×10^6 psi
ν=0.25

(a) Compute the unknown nodal displacements.
(b) Compute the stresses in the two elements.

Realize that this is a plane stress problem and therefore we need to use

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2 & 0.8 & 0 \\ 0.8 & 3.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \times 10^7 \ psi$$

<u>Step 1</u>: Node-element connectivity chart

ELEMENT	Node 1	Node 2	Node 3	Area (sqin)
1	1	2	4	3
2	3	4	2	3

Node	x	y
1	3	0
2	3	2
3	0	2
4	0	0

Nodal coordinates

<u>Step 2</u>: Compute strain-displacement matrices for the elements

Recall

$$\underline{B} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

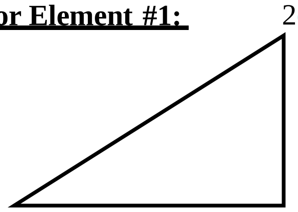
with

$$b_1 = y_2 - y_3 \quad b_2 = y_3 - y_1 \quad b_3 = y_1 - y_2$$

$$c_1 = x_3 - x_2 \quad c_2 = x_1 - x_3 \quad c_3 = x_2 - x_1$$

<u>For Element #1:</u>

2(2)



4(3) 1(1)

(local numbers within brackets)

$$y_1 = 0;\ y_2 = 2;\ y_3 = 0$$

$$x_1 = 3;\ x_2 = 3;\ x_3 = 0$$

Hence

$$b_1 = 2 \quad b_2 = 0 \quad b_3 = -2$$

$$c_1 = -3 \quad c_2 = 3 \quad c_3 = 0$$

Therefore

$$\underline{B}^{(1)} = \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

<u>For Element #2:</u>

$$\underline{B}^{(2)} = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & \end{bmatrix}$$

Step 3: **Compute element stiffness matrices**

$$\underline{k}^{(1)} = At\underline{B}^{(1)^{T}}\underline{D}\ \underline{B}^{(1)} = (3)(0.5)\underline{B}^{(1)^{T}}\underline{D}\ \underline{B}^{(1)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^{7}$$

$$\begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{array}$$

$$\underline{k}^{(2)} = At\underline{B}^{(2)^{T}}\underline{D}\ \underline{B}^{(2)} = (3)(0.5)\underline{B}^{(2)^{T}}\underline{D}\ \underline{B}^{(2)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^{7}$$

$$\begin{array}{cccccc} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{array}$$

Step 5: Compute consistent nodal loads

$$\underline{f} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{2y} \end{Bmatrix}$$

$$f_{2y} = -1000 + f_{S_{2y}}$$

The consistent nodal load due to traction on the edge 3-2

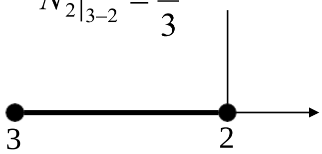
$$f_{S_{2y}} = \int_{x=0}^{3} N_3\big|_{3-2} (-300) t\,dx$$

$$= (-300)(0.5)\int_{x=0}^{3} N_3\big|_{3-2}\,dx$$

$$= -150\int_{x=0}^{3} \frac{x}{3}\,dx$$

$$= -50\left[\frac{x^2}{2}\right]_0^3 = -50\left(\frac{9}{2}\right) = -225\ lb$$

$$N_2\big|_{3-2} = \frac{x}{3}$$



Hence

$$f_{2y} = -1000 + f_{S_{2y}}$$
$$= -1225\ lb$$

Step 6: Solve the system equations to obtain the unknown nodal loads

$$\underline{K}\underline{d} = \underline{f}$$

$$10^7 \times \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1225 \end{Bmatrix}$$

Solve to get

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0.2337 \times 10^{-4}\ in \\ 0.1069 \times 10^{-4}\ in \\ -0.9084 \times 10^{-4}\ in \end{Bmatrix}$$

$\lfloor \;_2 \rfloor \quad \lfloor \qquad\qquad\qquad \rfloor$

Step 7: Compute the stresses in the elements

In Element #1

$$\underline{\sigma}^{(1)} = \underline{D}\,\underline{B}^{(1)}\underline{d}^{(1)}$$

With

$$\underline{d}^{(1)^T} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2337\times10^{-4} & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} & 0 & 0 \end{bmatrix}$$

Calculate

$$\underline{\sigma}^{(1)} = \begin{bmatrix} -114.1 \\ -1391.1 \\ -76.1 \end{bmatrix} psi$$

In Element #2

$$\underline{\sigma}^{(2)} = \underline{D}\,\underline{B}^{(2)}\underline{d}^{(2)}$$

With

$$\underline{d}^{(2)^T} = \begin{bmatrix} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} \end{bmatrix}$$

Calculate

$$\underline{\sigma}^{(2)} = \begin{bmatrix} 114.1 \\ 28.52 \\ -363.35 \end{bmatrix} psi$$

Notice that the stresses are constant in each element

Axi-Symmetric Models

Elasticity Equations

Elasticity equations are used for solving structural mechanics problems. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained. The types of elasticity equations are

➢ 1. Strian – Displacement relationship equations

$$e_x = \frac{\partial u}{\partial x};\; e_y = \frac{\partial v}{\partial y};\; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};\; \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x};$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

e_x – Strain in X direction, e_y – Strain in Y direction.

γ_{xy} - Shear Strain in XY plane, γ_{xz} - Shear Strain in XZ plane,

γ_{yz} - Shear Strain in YZ plane

2. Sterss – Strain relationship equation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{} \end{bmatrix} \begin{Bmatrix} e_x \\ e_x \\ e_x \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

σ – Stress, τ – Shear Stress, E – Young's Modulus, v – Poisson's Ratio,

e – Strain, γ - Shear Strain.

3. Equilibrium equations

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+B_x=0\,;\quad \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+\frac{\partial\tau_{xy}}{\partial x}+B_y=0$$

$$\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+B_z=0$$

σ – Stress, τ – Shear Stress, B_x - Body force at X direction,

B_y - Body force at Y direction, B_z - Body force at Z direction.

4. Compatibility equations

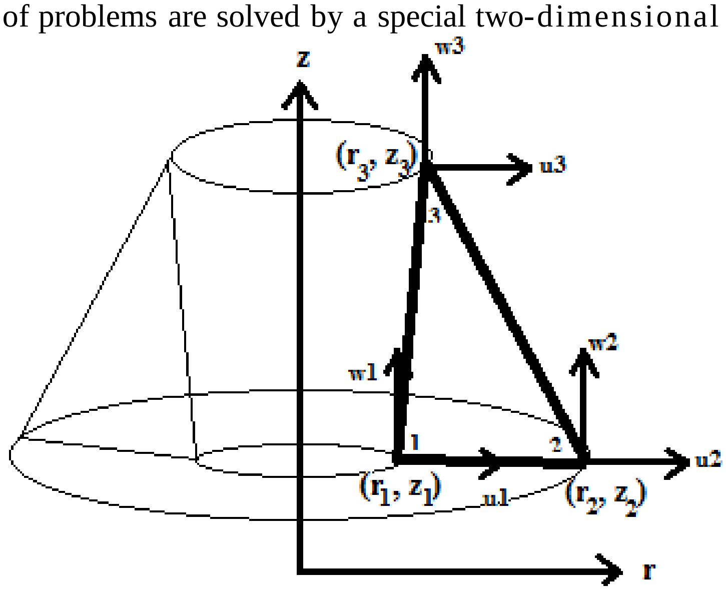
There are six independent compatibility equations, one of which is

$$\frac{\partial^2 e_x}{\partial y^2}+\frac{\partial^2 e_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}.$$

The other five equations are similarly second order relations.

- **Axisymmetric Elements**

Most of the three-dimensional problems are symmetry about an axis of rotation. Those types of problems are solved by a special two-dimensional element



calledas axisymmetric element.

- **Axisymmetric Formulation**

The displacement vector u is given by

$$u(r,z)=\begin{Bmatrix} u \\ w \end{Bmatrix}$$

The stress σ is given by

$$Stress, \{\sigma\}=\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

The strain e is given by

$$Strain, \{e\}=\begin{Bmatrix} e_r \\ e^\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

- **Equation of shape function for Axisymmetric element**

Shape function,

$$N_1=\frac{\alpha_1+\beta_1 r+\gamma_1 z}{2A};\ N_2=\frac{\alpha_2+\beta_2 r+\gamma_2 z}{2A};\ N_3=\frac{\alpha_3+\beta_3 r+\gamma_3 z}{2A}$$

$\alpha_1 = r_2z_3 - r_3z_2$; $\alpha_2 = r_3z_1 - r_1z_3$; $\alpha_3 = r_1z_2 - r_2z_1$

$\beta_1 = z_2 - z_3$; $\beta_2 = z_3 - z_1$; $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$; $\gamma_2 = r_1 - r_3$; $\gamma_3 = r_2 - r_1$

$2A = (r_2z_3 - r_3z_2) - r_1(r_3z_1 - r_1z_3) + z_1(r_1z_2 - r_2z_1)$

- **Equation of Strain – Displacement Matrix [B] for Axisymmetric element**

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$r = \frac{r1 + r2 + r3}{3}$$

- **Equation of Stress – Strain Matrix [D] for Axisymmetric element**

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

- **Equation of Stiffness Matrix [K] for Axisymmetric element**

$$[K] = 2\prod rA[B]^T[D][B]$$

$r = \dfrac{r1 + r2 + r3}{3}$; A = (½) bxh

- **Temperature Effects**

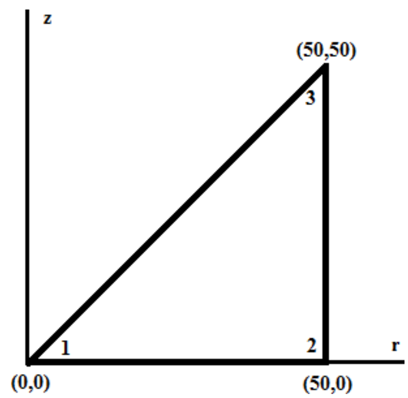
The thermal force vector is given by

$$\{f\}_t = 2\prod rA[B][D]\{e\}_t$$

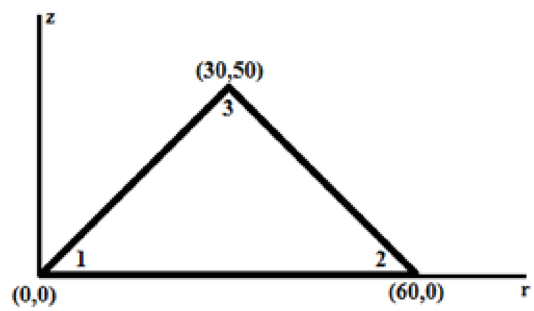
$$\{f\}_t = \begin{Bmatrix} F_1u \\ F_1w \\ F_2u \\ F_2w \\ F_3u \\ F_3w \end{Bmatrix}$$

- **Problem (I set)**

1. For the given element, determine the stiffness matrix. Take E=200GPa and υ= 0.25.



2. For the figure, determine the element stresses. Take $E=2.1\times10^5 N/mm^2$ and υ= 0.25. The co – ordinates are in mm. The nodal displacements are u_1=0.05mm, w_1=0.03mm, u_2=0.02mm, w_2=0.02mm, u_3=0.0mm, w_3=0.0mm.



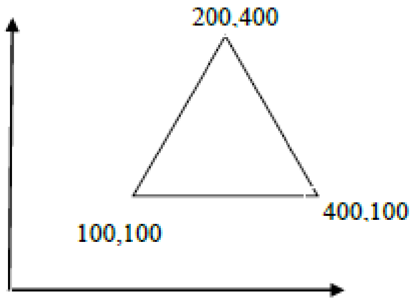
3. A long hollow cylinder of inside diameter 100mm and outside diameter 140mm is subjected to an internal pressure of 4N/mm2. By using two elements on the 15mm length, calculate the displacements at the inner radius.

UNIT III
Tutorial Questions

1. Derive the shape functions for CST Element

2. Derive the strain displacement matrix for CST element.

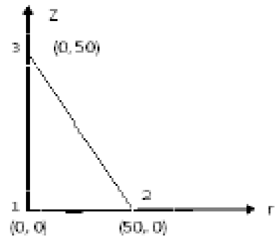
3. For the plane stress element shown in figure the nodal displacements are U1= 2.0mm, V1=1.0mm U2= 1.0 mm,V2= 1.5mm, U3= 2.5mm,V3=0.5mm, Take E= 210GPa, v= 0.25, t=10mm. Determine the strain-Displacement matrix [B]



Assignment Questions

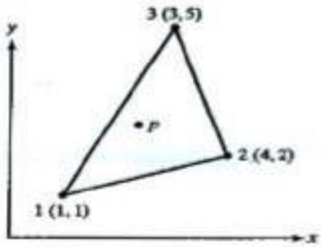
1. For axisymmetric element shown in figure, determine the strain-displacement matrix.

   Let $E = 2.1 \times 10^5 N/mm^2$ and $v = 0.25$. The co-ordinates shown in figure are in millimeters.



2.

a) Write the difference between CST and LST elements
b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P.



3. Derive the shape functions for axisymmetric element

UNIT 4

ISO-PARAMETRIC FORMULATION
&
HEAT TRANSFER PROBLEMS

Syllabus

Iso-Parametric Formulation: Concepts, sub parametric, super parametric elements, 2 dimensional 4 nodded iso-parametric elements, and numerical integration. Heat Transfer Problems: One dimensional steady state analysis composite wall. One dimensional fin analysis and two-dimensional analysis of thin plate.

OBJECTIVE:

To learn the application of FEM equations for Iso-Parametric and heat transfer problems

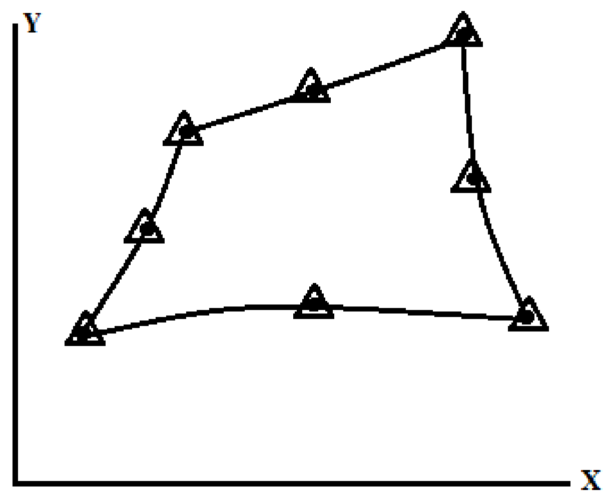
OUTCOME:

Formulate FE Characteristic equations for Isoparamteric problems and heattransfer problem.

UNIT – IV
ISOPARAMETRIC ELEMENTS

Isoperimetric element

Generally, it is very difficult to represent the curved boundaries by straight edge elements. A large number of elements may be used to obtain reasonable resemblance between original body and the assemblage. In order to overcome this drawback, isoparametric elements are used.



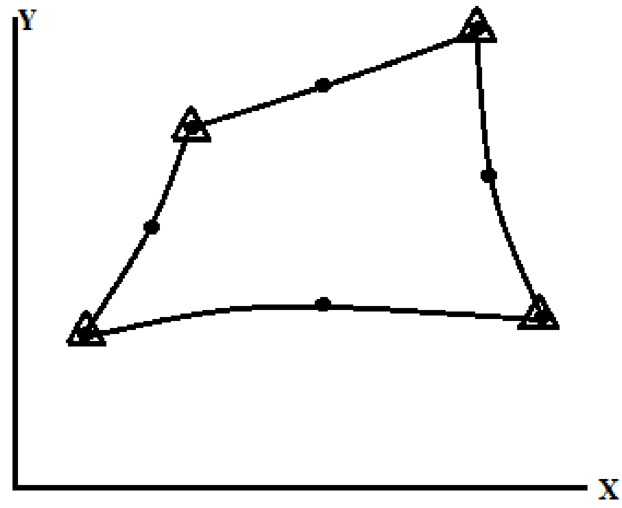
● — **Nodes used for defining geometry**

△ — **Nodes used for defining displacements**

If the number of nodes used for defining the geometry is same as number of nodes used defining the displacements, then it is known as isoparametric element.

- **Super parametric element**

If the number of nodes used for defining the geometry is more than number of nodes used for defining the displacements, then it is known as superparametric element.

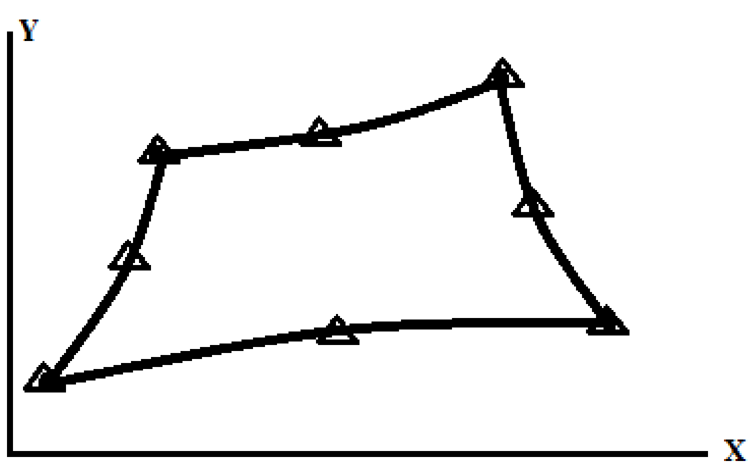


● — **Nodes used for defining geometry**

△ — **Nodes used for defining displacements**

- **Sub parametric element**

  If the number of nodes used for defining the geometry is less than number of nodes used for defining the displacements, then it is known as subparametric element.



● — **Nodes used for defining geometry**

△ — **Nodes used for defining displacements**

- **Equation of Shape function for 4 noded rectangular parent element**

$$u=\begin{Bmatrix} x \\ y \end{Bmatrix}=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} x_1 \\ y_2 \\ x_1 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

$N_1=1/4(1-\varepsilon)(1-\eta)$; $N_2=1/4(1+\varepsilon)(1-\eta)$; $N_3=1/4(1+\varepsilon)(1+\eta)$; $N_4=1/4(1-\varepsilon)(1+\eta)$.

- **Equation of Stiffness Matrix for 4 noded isoparametric quadrilateral element**

$$[K]=t\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]|J|\,\partial\varepsilon\,\partial\eta$$

$$[J]=\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix};$$

$$J_{11}=\frac{1}{4}\left[-(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4\right];$$

$$J_{12}=\frac{1}{4}\left[-(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4\right];$$

$$J_{21}=\frac{1}{4}\left[-(1-\varepsilon)x_1-(1+\varepsilon)x_2+(1+\varepsilon)x_3+(1-\varepsilon)x_4\right];$$

$$J_{22}=\frac{1}{4}\left[-(1-\varepsilon)y_1-(1+\varepsilon)y_2+(1+\varepsilon)y_3+(1-\varepsilon)y_4\right];$$

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \varepsilon} & 0 & \frac{\partial N_2}{\partial \varepsilon} & 0 & \frac{\partial N_3}{\partial \varepsilon} & 0 & \frac{\partial N_4}{\partial \varepsilon} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

$$[D] = \frac{E}{(1-v^2)}\begin{bmatrix} 1-v & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$, for plane stress conditions;

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$, for plane strain conditions.

- **Equation of element force vector**

$$\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix};$$

N – Shape function, F_x – load or force along x direction,

F_y – load or force along y direction.

- **Numerical Integration (Gaussian Quadrature)**

The Gauss quadrature is one of the numerical integration methods to calculate the definite integrals. In FEA, this Gauss quadrature method is mostly preferred. In this method the numerical integration is achieved by the following expression,

1 n

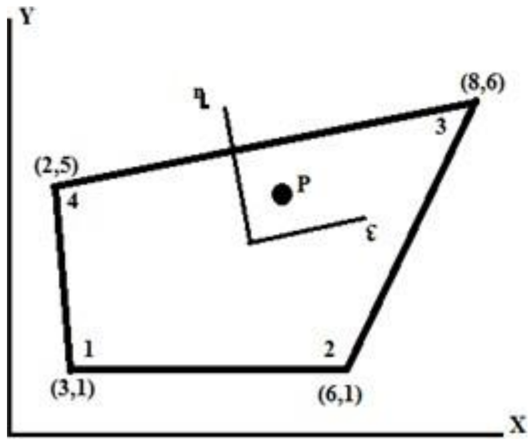
$$\int_{-1} f(x)dx = \sum_{i=1} w_i f(x_i)$$

Table gives gauss points for integration from -1 to 1.

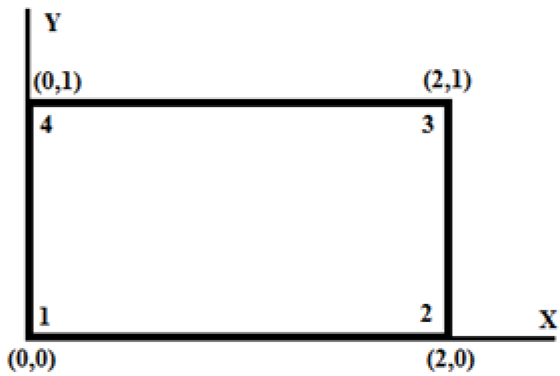
Number of Points n	Location x_i	Corresponding Weights w_i
1	**x_1 = 0.000**	**2.000**
2	**x_1, x_2 =** $\pm\sqrt{\frac{1}{3}} = \pm 0.577350269189$	**1.000**
3	**x_1, x_3** $= \pm\sqrt{\frac{3}{5}} = \pm 0.774596669241$ **x_2=0.000**	$\frac{5}{9} = 0.555555$ $\frac{8}{9} = 0.888888$
4	**x_1, x_4=** ± 0.8611363116 **x_2, x_3=** ± 0.3399810436	**0.3478548451** **0.6521451549**

- **Problem (I set)**

1. Evaluate $I = \int_{-1}^{1} \cos\frac{\pi x}{2} dx$, by applying 3 point Gaussian quadrature and compare with exact solution.
2. Evaluate $I = \int_{-1}^{1}\left[3e^x + x^2 + \frac{1}{x+2}\right]dx$, using one point and two point Gaussian quadrature. Compare with exact solution.
3. For the isoparametric quadrilateral element shown in figure, determine the local co –ordinates of the point P which has Cartesian co-ordinates (7, 4).



4. A four nodded rectangular element is in figure. Determine (i) Jacobian matrix, (ii) Strain – Displacement matrix and (iii) Element Stresses. Take $E=2x10^5 N/mm^2$, $\upsilon=0.25$, $u=[0,0,0.003,0.004,0.006, 0.004,0,0]^T$, $\varepsilon=0$, $\eta=0$. Assume plane stress condition.



Heat Transfer Problems

Heat transfer can be defined as the transmission of energy from one region to another region due to temperature difference. A knowledge of the temperature distribution within a body is important in many engineering problems. There are three modes of heat transfer.

They are: (*i*) Conduction
(*ii*) Convection
(*iii*) Radiation

(i) Conduction

Heat conduction is a mechanism of heat transfer from a region of high temperature to a region of low temperature within a medium (solid, liquid or gases) or between different medium in direct physical contact.

In conduction, energy exchange takes place by the kinematic motion or direct impact of molecules. Pure conduction is found only in solids.

(ii) Convection

Convection is a process of heat transfer that will occur between a solid surface and a fluid medium when they are at different temperatures.

Convection is possible only in the presence of fluid medium.

(iii) Radiation

The heat transfer from one body to another without any transmitting medium is known as radiation. It is an electromagnetic wave phenomenon.

. DERIVATION OF TEMPERATURE FUNCTION (T) AND SHAPE FUNCTION (N) FOR ONE DIMENSIONAL HEAT CONDUCTION ELEMENT

Consider a bar element with nodes 1 and 2 as shown in Fig. . T_1 and T_2 are the temperatures at the respective nodes. So, T_1 and T_2 are considered as degrees of freedom of this bar element.

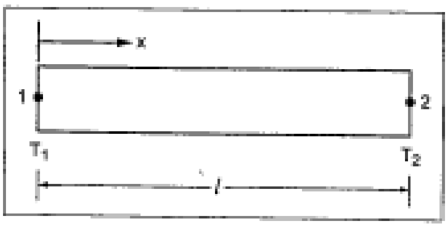


Fig.

Since the element has got two degrees of freedom, it will have two generalized co-ordinates.

$$\Rightarrow \qquad T = a_0 + a_1 x$$

where, a_0 and a_1 are global or generalized co-ordinates.

Writing the equation in matrix form,

$$T = [1 \quad x] \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

At node 1, $T = T_1, \quad x = 0$

At node 2, $T = T_2, \quad x = l$

Substitute the above values in equation

$\Rightarrow \quad T_1 = a_0$

$\Rightarrow \quad T_2 = a_0 + a_1 l$

Assembling the equations

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l-0} \begin{bmatrix} l & -0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\left[\text{Note: } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11} a_{22} - a_{12} a_{21})} \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$\Rightarrow \quad \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ values in equation

$$\Rightarrow \quad T = [1 \quad x] \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l} [1 \quad x] \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l} [l - x \quad 0 + x] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$[\because$ Matrix multiplication $(1 \times 2) \times (2 \times 2) = (1 \times 2)]$

$$T = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$T = [N_1 \quad N_2] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Temperature function, $T = N_1 T_1 + N_2 T_2$

where, Shape functions, $N_1 = \frac{l-x}{l}$

$N_2 = \frac{x}{l}$

DERIVATION OF STIFFNESS MATRIX FOR ONE DIMENSIONAL HEAT CONDUCTION ELEMENT

Consider a one dimensional bar element with nodes 1 and 2 as shown in Fig..... Let T_1 and T_2 be the temperatures at the respective nodes and k be the thermal conductivity of the material.

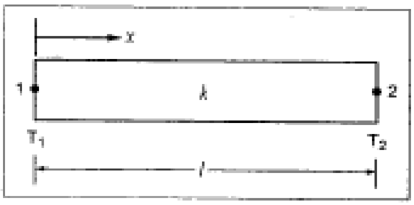


Fig.

We know that,

$$\text{Stiffness matrix } [K] = \int_v [B]^T [D][B]\, dv$$

In one dimensional element,

$$\text{Temperature function, } T = N_1 T_1 + N_2 T_2$$

$$\text{where, } N_1 = \frac{l - x}{l}$$

$$N_2 = \frac{x}{l}$$

We know that,

$$\text{Strain–Displacement matrix, } [B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$\Rightarrow \quad [B] = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$

$$\Rightarrow \quad [B]^T = \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix}$$

In one dimensional heat conduction problems,

$$[D] = [K] = k = \text{Thermal conductivity of the material}$$

Substitute $[B]$, $[B]^T$ and $[D]$ values in stiffness matrix equation

$$\Rightarrow \left.\begin{matrix}\text{Stiffness matrix for}\\ \text{heat conduction}\end{matrix}\right\} [K_C] = \int_0^l \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \times k \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} k\ dv$$

[Matrix multiplication $(2 \times 1) \times (1 \times 2) = (2 \times 2)$]

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} k A\ dx \qquad [\because dv = A \times dx]$$

$$= A k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^l dx$$

$$= A k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} [x]_0^l$$

$$= A k \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} (l - 0)$$

$$= A k l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix}$$

$$= \frac{A k l}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_C] = \frac{A k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where, A = Area of the element, m^2

k = Thermal conductivity of the element, W/mK

l = Length of the element, m

FINITE ELEMENT EQUATIONS FOR ONE DIMENSIONAL HEAT CONDUCTION PROBLEMS

We know that,

General force equation is, $\{F\} = [K_C]\{T\}$

where, $\{F\}$ is a element force vector [Column matrix]

$[K_C]$ is a stiffness matrix [Row matrix]

$\{T\}$ is a nodal temperature [Column matrix]

For one dimensional heat conduction problems, stiffness matrix $[K]$ is given by

$$[K_C] = \frac{A k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Consider a two noded element as shown in Fig.2.36.

Force vector $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$

Nodal temperature $\{T\} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$

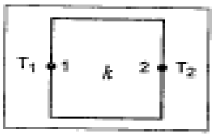


Fig. .

Substitute $[K_C]\{F\}$ and $\{T\}$ values in equation

$$\Rightarrow \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Case (i): One dimensional heat conduction with free end convection

Consider a one dimensional element with nodes 1 and 2 as shown in Fig T_1 and T_2 are the temperatures at the respective nodes. Assume convection occurs only from the right end of the element as shown in Fig.

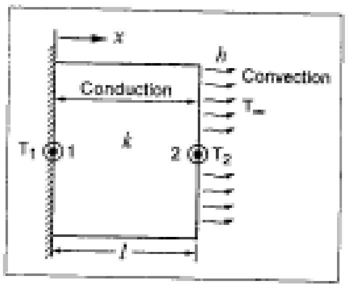


Fig. .

Stiffness matrix $[K_C]$ for one dimensional heat conduction element is given by

$$[K_C] = \frac{A k}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The convection term contribution to the stiffness matrix is given by

$$[K_h]_{end} = \iint_A h [N]^T [N] \, dA$$

where, h = Heat transfer coefficient, W/m²K

N = Shape factor

We know that,

$$\text{Shape factor, } [N] = [N_1 \;\; N_2] = \left[\frac{l-x}{l} \;\; \frac{x}{l}\right]$$

At node 2, $\qquad x = l$

$$\Rightarrow \qquad [N] = [N_1 \;\; N_2] = [0 \;\; 1]$$

$$\Rightarrow \qquad [N]^T = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Substitute [N] and $[N]^T$ values in equation

$$\Rightarrow \qquad [K_h]_{end} = \iint_A h \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} [0 \;\; 1] \, dA$$

$$[(2 \times 1) \times (1 \times 2) = (2 \times 2)]$$

$$[K_h]_{end} = h \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \int dA$$

$$[K_h]_{end} = h A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \ldots.$$

$$\text{Stiffness matrix } [K] = [K_C] + [K_h]$$

$$\Rightarrow \qquad [K] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The convection force from the free end of the element is obtained from the following relation,

$$\{F_h\}_{end} = h T_\infty A \begin{Bmatrix} N_1 \; (x = l) \\ N_2 \; (x = l) \end{Bmatrix}$$

$$\{F_h\}_{end} = h T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

We know that, General force equation is

$$\{F\} = [K]\{T\}$$

Substitute { F } and [K] values,

$$\Rightarrow \qquad h T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left[\frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\Rightarrow \left[\frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = h T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

where, A = Area of the element, m^2
k = Thermal conductivity of the element, W/mK
l = Length of the element
h = Heat transfer coefficient, W/m^2K
T_∞ = Fluid temperature, K
T = Temperature, K

This is a finite element equation for one dimensional heat conduction element with free end convection.

Case (ii): One dimensional element with conduction, convection and internal heat generation:

Consider a rod with nodes 1 and 2 as shown in Fig. . This rod is subjected to conduction, convection and internal heat generation.

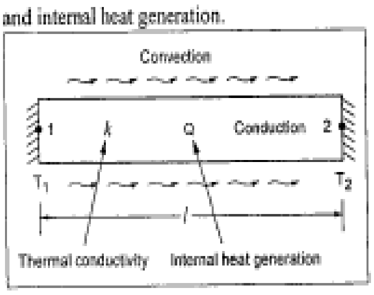


Fig.

We know that, heat conduction part of the stiffness matrix [K] for the one dimensional element is

$$[K_C] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad ....$$

Heat convection part of the stiffness matrix [K] for the one dimensional element is given by

$$[K_h] = \iint_S h[N]^T [N]\, dS$$

$$= hP\int_0^l [N]^T [N]\, dx$$

$[\because dS = P \times dx$ where P = Perimeter of the element$]$

$$= hP\int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} dx$$

$$= hP\int_0^l \begin{bmatrix} \left(\frac{l-x}{l}\right)^2 & \left(\frac{l-x}{l}\right)\times\frac{x}{l} \\ \frac{x}{l}\times\left(\frac{l-x}{l}\right) & \frac{x}{l}\times\frac{x}{l} \end{bmatrix} dx$$

$$= hP\int_0^l \begin{bmatrix} \left(1-\frac{x}{l}\right)^2 & \frac{x}{l}-\frac{x^2}{l^2} \\ \frac{x}{l}-\frac{x^2}{l^2} & \frac{x^2}{l^2} \end{bmatrix} dx$$

$$= hP\begin{bmatrix} \dfrac{\left(1-\frac{x}{l}\right)^3}{3\times\left(\frac{-1}{l}\right)} & \frac{x^2}{2l}-\frac{x^3}{3l^2} \\ \frac{x^2}{2l}-\frac{x^3}{3l^2} & \frac{x^3}{3l^2} \end{bmatrix}_0^l$$

$$= hP\begin{bmatrix} \dfrac{\left(1-\frac{l}{l}\right)^3}{\frac{-3}{l}} - \dfrac{1^3}{\frac{-3}{l}} & \frac{l^2}{2l}-\frac{l^3}{3l^2}-0 \\ \frac{l^2}{2l}-\frac{l^3}{3l^2}-0 & \frac{l^3}{3l^2}-0 \end{bmatrix}$$

$$= hP\begin{bmatrix} \frac{l}{3} & \frac{l}{2}-\frac{l}{3} \\ \frac{l}{2}-\frac{l}{3} & \frac{l}{3} \end{bmatrix}$$

$$= hP\begin{bmatrix} \frac{l}{3} & \frac{l}{6} \\ \frac{l}{6} & \frac{l}{3} \end{bmatrix}$$

$$[K_h] = \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Stiffness matrix $[K] = [K_C] + [K_h]$

$$\Rightarrow \quad [K] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Force matrix due to heat generation is given by,

$$\{F_Q\} = \iiint_V [N]^T \; Q \; dV$$

$$= \int_0^l [N]^T \times Q \times A \times dx \qquad [\because dV = A \times dx]$$

$$= Q \times A \int_0^l [N]^T \; dx$$

$$= Q \times A \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= Q \times A \int_0^l \begin{Bmatrix} 1 - \frac{x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= Q \times A \begin{Bmatrix} x - \frac{x^2}{2\,l} \\ \frac{x^2}{2\,l} \end{Bmatrix}_0^l$$

$$= Q \times A \begin{Bmatrix} l - \frac{l^2}{2\,l} - 0 \\ \frac{l^2}{2\,l} - 0 \end{Bmatrix}$$

$$= Q \times A \begin{Bmatrix} \frac{l^2}{2\,l} \\ \frac{l^2}{2\,l} \end{Bmatrix} = Q \times A \begin{Bmatrix} \frac{l}{2} \\ \frac{l}{2} \end{Bmatrix}$$

$$\{F_Q\} = Q \times A \times \frac{l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Force matrix due to convection is given by

$$\{F_h\} = \iint_S h\, T_\infty \; [N]^T \; dS$$

$$= \iint_S h \; T_\infty \; [N]^T \; P \times dx \qquad [\because dS = P \times dx]$$

$$= P\, h\, T_\infty \int_0^l [N]^T \; dx$$

$$= P\, h\, T_\infty \int_0^l \begin{Bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{Bmatrix} dx$$

$$= P\, h\, T_\infty \begin{Bmatrix} x - \frac{x^2}{2\,l} \\ \frac{x^2}{2\,l} \end{Bmatrix}_0^l$$

$$= P h T_\infty \begin{Bmatrix} l - \frac{l^2}{2l} - 0 \\ \frac{l^2}{2l} - 0 \end{Bmatrix}$$

$$= P h T_\infty \begin{Bmatrix} \frac{l}{2} \\ \frac{l}{2} \end{Bmatrix}$$

$$\{F_h\} = \frac{P h T_\infty l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Adding equations

$$\text{Force matrix, } \{F\} = \{F_Q\} + \{F_h\}$$

$$= \frac{QAl}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{P h T_\infty l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{F\} = \frac{QAl + P h T_\infty l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

We know that, General force equation is

$$\{F\} = [K]\{T\}$$

Substitute { F } and [K] values,

$$\Rightarrow \quad \frac{QAl + P h T_\infty l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \left[\frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\Rightarrow \quad \left[\frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{QAl + P h T_\infty l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

where,
- A = Area of the element, m^2
- k = Thermal conductivity of the element, W/mK
- l = Length of the element, m
- h = Heat transfer coefficient, W/m^2K
- P = Perimeter, m
- T = Temperature, K
- Q = Heat generation, W
- T_∞ = Fluid temperature, K

This is a finite element equation for one dimensional element which is subjected to conduction, convection and internal heat generation.

Example *A wall of 0.6 m thickness having thermal conductivity of 1.2 W/mK. The wall is to be insulated with a material of thickness 0.06 m having an average thermal conductivity of 0.3 W/mK. The inner surface temperature is 1000°C and outside of the insulation is exposed to atmospheric air at 30°C with heat transfer coefficient of 35 W/m²K. Calculate the nodal temperatures.*

Given: Thickness of the wall, $l_1 = 0.6$ m

Thermal conductivity of the wall, $k_1 = 1.2$ W/mK

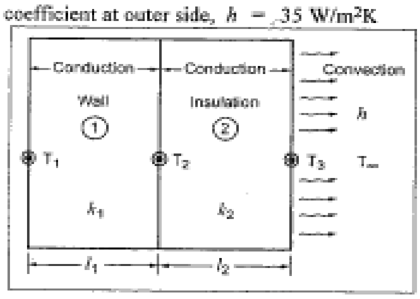
Thickness of the insulation, $l_2 = 0.06$ m

Thermal conductivity of the insulation, $k_2 = 0.3$ W/mK

Inner surface temperature, $T_1 = 1000°C + 273 = 1273$ K

Atmospheric air temperature, $T_\infty = 30°C + 273 = 303$ K

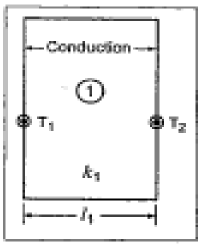
Heat transfer coefficient at outer side, $h = 35$ W/m²K



To find: Nodal temperatures, (T_2 and T_3)

Solution:

For element 1: (Nodes 1, 2)



Finite element equation is

$$\frac{A_1 k_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

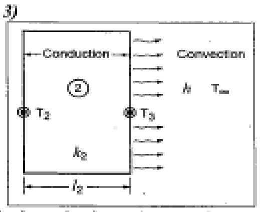
[From equation no.(2.131)]

For unit area, $A_1 = 1\ \text{m}^2$

$$\frac{1.2}{0.6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \ldots (1)$$

For element 2: (Nodes 2, 3)



This element is subjected to both conduction and convection. So, finite element equation is

$$\left(\frac{A_2 k_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = hT_\infty A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\frac{1 \times 0.3}{0.06}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 35 \times 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = 35 \times 303 \times 1\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 35 \end{bmatrix}\right)\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

$$\begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 5 & -5 \\ -5 & 40 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10.605 \times 10^3 \end{Bmatrix} \quad \ldots (2)$$

Assemble the finite elements, *i.e.*, assemble the finite element equations (1) and (2).

$$\begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2+5 & -5 \\ 0 & -5 & 40 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

In this problem, there is no heat generation and there is no convection except from right end.

So, $\{F_1\} = \{F_2\} = 0$ and $\{F_3\} = 10.605 \times 10^3$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 2 & -2 & 0 \\ -2 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix}}_{[K]}\underbrace{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}}_{\{T\}} = \underbrace{\begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}}_{\{F\}} \quad \ldots (3)$$

To solve the above equation, the following steps to be followed.

Step 1: The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 2: The first row of the force matrix is replaced by the known temperature at node 1, *i.e.*, T_1.

$$\Rightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 3: The second row, first column of stiffness matrix [K] value (From equation no.3) is multiplied by known temperature at node 1, *i.e.*, $-2 \times 1273 = -2546$. This value (as positive digit, *i.e.*, 2546) has been added to the second row of the force matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 2546 \\ 10.605 \times 10^3 \end{Bmatrix} \qquad \ldots (4)$$

Solving equation (4),

$$\Rightarrow \qquad 7\,T_2 - 5\,T_3 = 2546 \qquad \ldots (5)$$

$$-5\,T_2 + 40\,T_3 = 10.605 \times 10^3 \qquad \ldots (6)$$

Equation (5) × 8,

$$\Rightarrow \qquad 56\,T_2 - 40\,T_3 = 20.368 \times 10^3 \qquad \ldots (7)$$

Equation (6) ⇒ $\quad -5\,T_2 + 40\,T_3 = 10.605 \times 10^3$

$$51\,T_2 = 30.973 \times 10^3$$

$$\Rightarrow \qquad \boxed{T_2 = 607.313 \text{ K}}$$

Substitute T_2 value in equation (5),

$$\Rightarrow \qquad 7 \times 607.313 - 5\,T_3 = 2546$$

$$\Rightarrow \qquad \boxed{T_3 = 341.03 \text{ K}}$$

Result: ***Nodal temperatures:*** $T_1 = 1273$ K

$T_2 = 607.313$ K

$T_3 = 341.03$ K

UNIT IV
Tutorial Questions

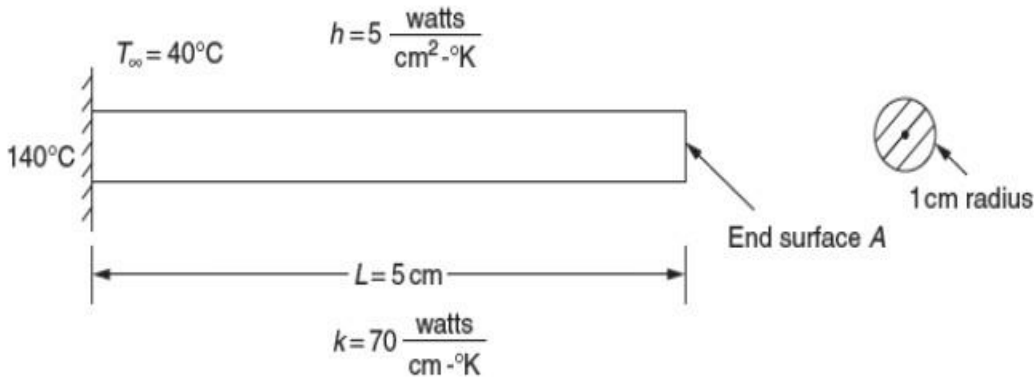
1. A composite slab consists of 3 materials of different conductivities i.e. 22 W/m K, 32 W/m K, 52 W/m K of thickness 0.31 m, 0.14 m and 0.14 m, respectively. The outer surface is 22^0 C and the inner surface is exposed to the convective heat transfer coefficient of 28 W/m^2 K, 800^0 C. Determine the temperature distribution within the wall

.

2. Determine the temperature distribution along a circular fin of length 5 cm and radius1 cm. The fin is attached to boiler whose wall temperature 1400C and the free end is open to the atmosphere. Assume T_∞= 40^0C, h = 10 W/cm^2 /^{0} C, k = 70 W/cm ^{0}C.

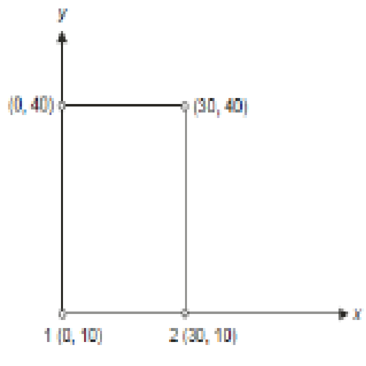
3. Discuss in detail about 2D heat conduction in Composite slabs using FEA

4. Find the temperature distribution in the one-dimensional fin shown in Figure below using two finite elements.



.

5. Consider a quadrilateral element as shown in figure, Evaluate Jacobian matrix and strain-Displacement matrix at local coordinates ξ =0.5, η = 0.5.



Assignment Questions

1.

Evaluate the integral $\int_{-1}^{+1}\left[3e^{x}+2x^{2}+\frac{1}{(3x+4)}\right]dx$ using one point and two point Gauss quadrature.

2. Evaluate the following integral using Gaussian quadrature, so that the result is exact.

$$f(r)=\int_{-1}^{1}\left(\frac{1}{1+x^{2}}+2x-\sin x\right)dx$$

3. Estimate the temperature distribution in a fin whose cross section is 15mm X 15mm and 500mm long. Take Thermal conductivity as 50W/m-k and convective heat transfer coefficient as 75 W/m^2-k at 25^{o}C. The base temperature is assumed to be constant and its value may be taken as 900^{o}C. And also calculate the heat transfer rate?

UNIT 5

DYNAMIC ANALYSIS

Syllabus

Dynamic Analysis: Formulation of finite element model, element matrices, evaluation of Eigen values and Eigen vectors for a stepped bar and a beam.

OBJECTIVE:

To learn the application of FEM equations for dynamic analysis

OUTCOME:

Solve dynamic problems where the effect of mass matters during the analysis

UNIT-V
DYNAMIC ANALYSIS

Dynamics is a special branch of mechanics where inertia of accelerating masses must be considered in the force-deflection relationships. **In** order to describe motion of the mass system, a component with distributed mass is approximated by a finite number of mass points. Knowledge of certain principles of dynamics is essential to the formulation of these equations.

Every structure is associated with certain frequencies and mode shapes of free vibration (without continuous application of load), based on the distribution of mass and stiffness in the structure. Any time-dependent external load acting on the structure, whose frequency matches with the natural frequencies of the structure, causes resonance and produces large displacements leading to failure of the structure. Calculation of natural frequencies and mode shapes is there for every important.

In general, for a system with on' degrees of freedom, stiffness 'k' and mass 'm' are represented by stiffness matrix [K] and mass matrix [M] respectively.

Then

$$([K] - \omega^2 [M])\{u\} = \{0\}$$

$$([M]^{-1}[K] - \omega^2 [I])\{u\} = \{0\}$$

Here, [M] is the mass matrix of the entire structure and is of the same order, say n x n, as the stiffness matrix [K]. This is also obtained by assembling element mass matrices in a manner exactly identical to assembling element stiffness matrices. The mass matrix is obtained by two different approaches, as explained subsequently.

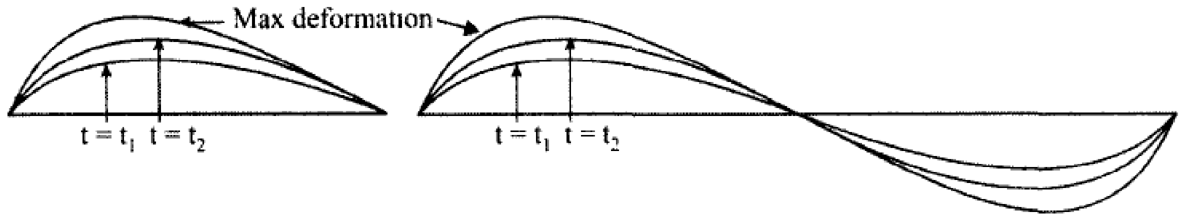
A structure with 'n' DOF will therefore have 'n' eigen values and 'n' eigenvectors. Some eigen values may be repeated and some eigen values maybe complex, in pairs. The equation can be represented in the standard form,

$[A]\{x\}_i = \lambda_i\{x\}_i$.

In dynamic analysis, ω_i, indicates ith natural frequency and $\{X\}_I$ indicates ith natural mode of vibration.

A natural mode is a *qualitative* plot of nodal displacements. In every natural mode of vibration, all the points on the component will reach their maximum values at the same time and will pass through zero displacements at the same time. Thus, in a particular mode, all the points of a component will vibrate with the same frequency and their relative displacements are indicated by the components of the corresponding eigen vector. These relative (or proportional) displacements at different points

on structure remain same at every time instant for undamped free vibration.



Hence, without loss of generality, {u(t)} can be written as {u}.

Since {u} = {O} forms a trivial solution, the homogeneous system of equations

([A] - λ[I]) {u} = {O}

gives a non-trivial solution only when

([A]-λ[I])= {O},

which implies

Det([A] - λ[I]) = o.

This expression, called *characteristic equation,* results in nth order polynomial in A, and will therefore have n roots. For each λ, the corresponding eigenvector {u}. can be obtained from the n homogeneous equations represented by

([K] - λ[M]) {u} = {O}.

The mode shape represented by {u(t)}gives relatives values of displacements in various degrees of freedom.

NORMALIZATION

The equation of motion of free vibrations ([K] $-\omega^2$[M]) {u} = {O} is a system of homogeneous equations (right side vector zero) and hence does not give unique numerical solution.

Mode shape is a set of relative displacements in various degrees of freedom, while the structure is vibrating in a particular frequency and is usually expressed in normalized form, by following one of the

three normalization methods explained here.

(a) The maximum value of anyone component of the eigenvector is equated to 'I' and, so, all other components will have a value less than or equal to ' 1' .

(b) The length of the vector is equated to '1 ' and values of all components are divided by the length of this vector so that each component will have a value less than or equal to '1'.

(c) The eigenvectors are usually normalized so that

$$\{u\}_i^T [M] \{u\}_i = 1 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_i = \lambda_i$$

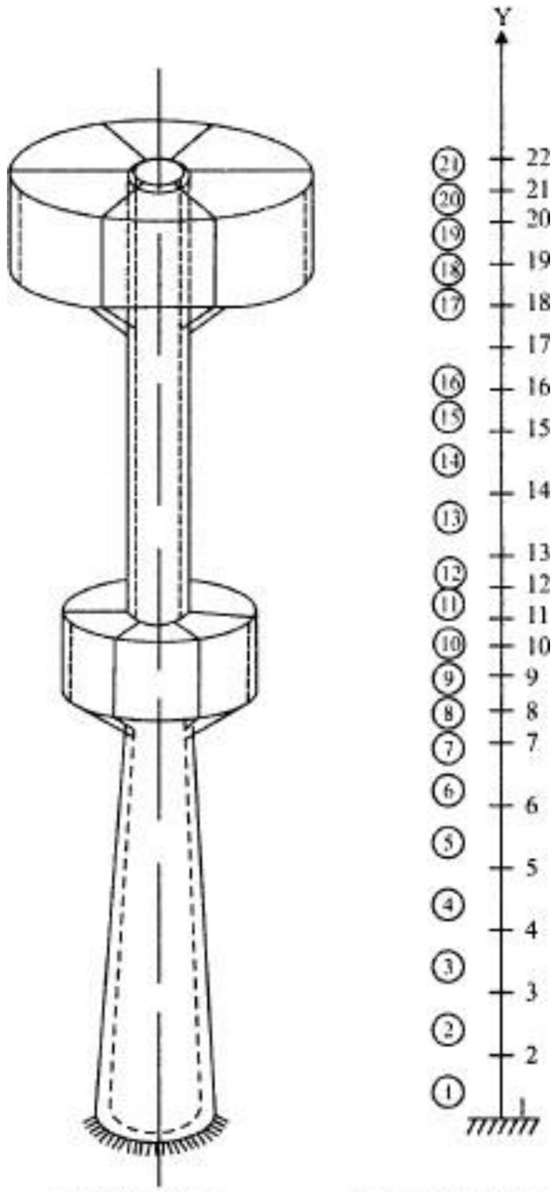
For a positive definite symmetric stiffness matrix of size n x n, the Eigen values are all real and eigenvectors are orthogonal

i.e.,

$$\{u\}_i^T [M] \{u\}_j = 0 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_j = 0 \quad \forall \; i \neq j$$

MODELLING FOR DYNAMIC ANALYSIS

Solution for any dynamic analysis is an iterative process and, hence, is time -consuming. Geometric model of the structure for dynamic analysis can be significantly simplified, giving higher priority for proper representation of distributed mass. An example of a simplified model of a water storage tank is shown in Fig. Below, representing the central hollow shaft by long beam elements and watertanks at two levels by a few lumped masses and short beam elements of larger moment of inertia.



Actual structure Idealised structure

MASS MATRIX

Mass matrix [M] differs from the stiffness matrix in many ways:

(i) The mass of each element is equally distributed at all the nodes of that element

(ii) Mass, being a scalar quantity, has same effect along the three translational degrees of freedom (u, v and w) and is not shared

(iii) Mass, being a scalar quantity, is not influenced by the local or globa coordinate system. Hence, no transformation matrix is used for converting mass matrix from element (or local) coordinate system to structural (or global) coordinate system.

Two different approaches of evaluating mass matrix [M] are commonly considered.

(a) Lumped mass matrix

Total mass of the element is assumed equally distributed at all the nodes of the element in each of the translational degrees of freedom. Lumped mass is not used for rotational degrees of freedom. Off-diagonal elements of this matrix are all zero. This assumption *excludes dynamic coupling* that exists between different nodal displacements.

Lumped mass matrices [M] of some elements are given here.

Lumped mass matrix of truss element with 1 translational DOF per node along its local X-axis

$$[M]= \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lumped mass matrix of plane truss element in a 2-D plane with2 translational DOF per node (Displacements along X and Y coordinate axes)

$$[M]= \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Please note that the same lumped mass is considered in each translational degree of freedom (without proportional sharing of mass between them) at each node.

Lumped mass matrix of a beam element in X-V plane, with its axis along x-axis and with two DOF per node (deflection along Y axis and slope about Z axis) is given below. Lumped mass is not considered in the rotational degrees of freedom.

$$[M]= \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that lumped mass terms are not included in 2nd and 4th rows, as well as columns corresponding to rotational degrees of freedom.

Lumped mass matrix of a CST element with 2 DOF per node. In this case, irrespective of the shape of the element, mass is assumed equally distributed at the three nodes. It is distributed equally in all DOF at each node, without any sharing of mass between different DOF

$$[M]= \frac{\rho AL}{3}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Consistent mass matrix

Element mass matrix is calculated here, *consistent* with the assumed displacement field or element stiffness matrix. [M] is a banded matrix of the same order as the stiffness matrix. This is evaluated using the same

interpolating functions which are used for approximating displacement field over the element. It yields more accurate results but with more computational cost. Consistent mass matrices of some elements are given here.

Consistent mass matrix of a Truss element along its axis (in local coordinate system)

$$\{u\}^T = [u \quad v]$$

$$[N]^T = [N_1 \quad N_2]$$

where, $N_1 = \frac{(1-\xi)}{2}$

and $N_2 = \frac{(1+\xi)}{2}$

$$[M] = \int_v [N]\rho[N]^T \, dV = \int_0^L A[N]\rho[N]^T$$

$$dx = \int_{-1}^{+1} A\,\rho\,[N][N]^T\,(\det J)(dx/d\xi)d\xi$$

Here, $x = N_1 x_1 + N_2 x_2 = \frac{(x_1 + x_2)}{2} + \frac{(x_2 - x_1)\xi}{2}$

and $dx = \frac{dx}{d\xi}.d\xi = \det J \, d\xi = \left(\frac{L}{2}\right) d\xi$

Using the values of integration in natural coordinate system,

$$[M]=\rho A\left(\frac{L}{2}\right)\int_{-1}^{+1}\begin{bmatrix}(1-\xi)/2\\(1+\xi)/2\end{bmatrix}\left[(1-\xi)/2 \quad (1+\xi)/2\right]d\xi$$

$$=\frac{\rho AL}{8}\begin{bmatrix}\int(1-\xi)^2 d\xi & \int(1-\xi^2)d\xi\\ \int(1-\xi^2)d\xi & \int(1+\xi)^2 d\xi\end{bmatrix}$$

$$=\frac{\rho AL}{8}\begin{bmatrix}(\xi-\xi^2+\xi^3/3) & (\xi-\xi^3/3)\\ (\xi-\xi^3/3) & (\xi+\xi^2+\xi^3/3)\end{bmatrix}$$

$$=\frac{\rho AL}{8}\begin{bmatrix}8/3 & 4/3\\ 4/3 & 8/3\end{bmatrix}=\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Consistent mass *matrix (if a* **Plane** *Truss element,* inclined to global X-axis -Same elements of I-D mass matrix are repeated in two dimensions (along X and Y directions) without sharing mass between them. Mass terms in X and Y directions are uncoupled.

$$[M]=\frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 2\end{bmatrix}$$

Consistent mass matrix of a Space *Truss element,* inclined to X-Y plane) -Same elements of 1-0 mass matrix are repeated in three dimensions (along X, Y and Z directions) without sharing mass between them.

$$[M]=\frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 0 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 0 & 1\\ 1 & 0 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 0\\ 0 & 0 & 1 & 0 & 0 & 2\end{bmatrix}$$

Consistent *mass matrix of a Beam element*

$[M] = \rho A\left(\frac{L}{2}\right)\int \{H\}^T\{H\}d\xi$ with Hermite shape functions $\{H\}$ as used in a beam element.

$$= \frac{\rho AL}{128}\int\begin{bmatrix} 2\left(2-3\xi+\xi^3\right) \\ L\left(1-\xi+\xi^2+\xi^3\right) \\ 2\left(2+3\xi-\xi^3\right) \\ L\left(-1-\xi+\xi^2+\xi^3\right) \end{bmatrix} \times$$

$$\left[2\left(2-3\xi+\xi^3\right) \quad L\left(1-\xi-\xi^2+\xi^3\right) \quad 2\left(2+3\xi-\xi^3\right) \quad L\left(-1-\xi+\xi^2+\xi^3\right)\right]d\xi$$

$$= \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Consistent mass matrix of a CST element in a 2-D plane

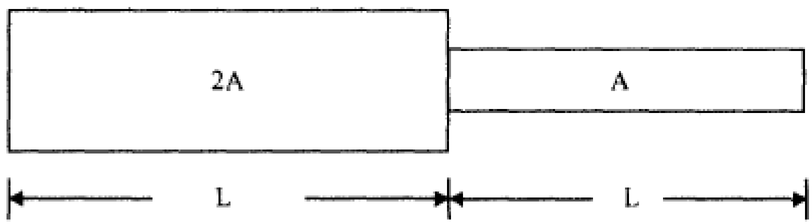
$$[N]^T = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$[M] = \int [N]\rho[N]^T\, dV = t\int [N]\rho[N]^T\, dA$$

$$= \frac{\rho tA}{12}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ & 2 & 0 & 1 & 0 & 1 \\ & & 2 & 0 & 1 & 0 \\ & & & 2 & 0 & 1 \\ & \text{Sym} & & & 2 & 0 \\ & & & & & 2 \end{bmatrix}$$

Note: Natural frequencies obtained using lumped mass matrix are LOWER than exact values.

Example 1 : Find the natural frequencies of longitudinal vibrations of the unconstrained stepped shaft of areas A and 2A and of equal lengths (L), as shown below.



Solution: Let the finite element model of the shaft be represented by 3 nodes and 2 truss elements (as only longitudinal vibrations are being considered) as shown below.

1. Longitudinal vibration of bar

Finite element equation,

$$\{[K]-[m]\omega^2\}\{u\} = \{F\}$$

where, Stiffness matrix, $[K] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Mass matrix, $[m] = \frac{\rho A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ for consistent mass matrix

$[m] = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for lumped mass matrix

2. Transverse vibration of beam

Finite element equation,

$$\{[K]-[m]\omega^2\}\{u\} = \{F\}$$

where, Stiffness matrix, $[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

Mass matrix, $[m] = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$

for consistent mass matrix

$[m] = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for lumped mass matrix

Example ***Find the natural frequency of longitudinal vibration of the unconstrained stepped bar as shown***

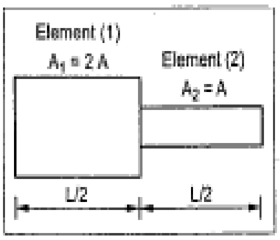


Fig. (i).

Given:

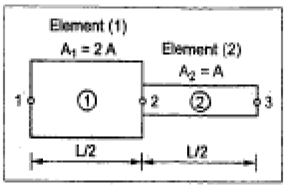


Fig. (ii).

Element (1)	**Element (2)**
Area, $A_1 = 2A$	Area, $A_2 = A$
Length, $L_1 = \frac{L}{2}$	Length, $L_2 = \frac{L}{2}$
Young's modulus, $E_1 = E$	Young's modulus, $E_2 = E$
Density, $\rho_1 = \rho$	Density, $\rho_2 = \rho$

To find: Natural frequencies of the rod.

Solution: The bar with two element and 3 nodes are as shown in Fig.*(ii)*. The stiffness matrix of the two elements are,

Stiffness matrix for Element (1),

$$[K_1] = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2AE}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{4AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_1] = \frac{2AE}{L}\begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \end{array}$$

Similarly, Stiffness matrix for Element (2),

$$[K_2] = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{AE}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_2] = \frac{2AE}{L}\begin{array}{c} \begin{array}{cc} \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{2} \\ \mathbf{3} \end{array} \quad \text{...(2)}$$

Assemble the stiffness matrix,

$$[K] = \frac{2AE}{L}\begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \quad \text{...(3)}$$

Mass Matrix for Element (1),

$$[m_1] = \frac{\rho_1 A_1 L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_1] = \frac{\rho \times 2A \times \frac{L}{2}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_1] = \frac{\rho A L}{12}\begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \quad \ldots (4)$$

Similarly, Mass matrix for Element (2),

$$[m_2] = \frac{\rho_2 A_2 L_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho A \times \left(\frac{L}{2}\right)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_2] = \frac{\rho A L}{12}\begin{array}{cc} 2 & 3 \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \quad \ldots (5)$$

Assemble the mass matrix,
$$[m] = \frac{\rho A L}{12}\begin{array}{cc} 1 \quad 2 \quad 3 & \\ \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \quad \ldots (6)$$

Since, the bar is unconstrained (no degrees of freedom is fixed), the finite element equation is

$$\{[K] - [m]\omega^2\}\{u\} = \{P\}$$

Substitute [K] and [m] values

$$\left[\frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2\frac{\rho A L}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Applying boundary conditions,

$$P_1 = P_2 = P_3 = 0$$

[No degrees of freedom is fixed]

We set the determinant of the coefficient matrix equal to zero, we have

$$\left|\frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2\frac{\rho A L}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0 \quad \ldots (7)$$

Divide both sides by $\left(\frac{2AE}{L}\right)$,

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\omega^2\frac{\rho A L}{12}}{\frac{2AE}{L}}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

Divide both sides by $\left(\frac{2AE}{L}\right)$,

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\omega^2\frac{\rho A L}{12}}{\frac{2AE}{L}}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\rho L^2\omega^2}{24E}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0 \quad \ldots (8)$$

Take, $\beta^2 = \frac{\rho L^2\omega^2}{24E}$

Equation (8) can be rewritten as,

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \beta^2\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 2(1-2\beta^2) & -2(1+\beta^2) & 0 \\ -2(1+\beta^2) & 3(1-2\beta^2) & -(1+\beta^2) \\ 0 & -(1+\beta^2) & (1-2\beta^2) \end{vmatrix} = 0$$

$$\Rightarrow 2(1-2\beta^2)[3(1-2\beta^2)^2 - (1+\beta^2)^2] + 2(1+\beta^2)[-2(1+\beta^2)(1-2\beta^2)] = 0$$

By simplifying the above equation, we get

$$\Rightarrow 18\,\beta^2(1-2\beta^2)(\beta^2-2) = 0 \qquad \dots(9)$$

The roots of equation (9) give the natural frequencies of the bar.

We know that, $\beta^2 = \dfrac{\rho L^2 \omega^2}{24 E}$

when, $\beta^2 = 0 \Rightarrow \omega_1^2 = 0 \Rightarrow \omega_1 = 0$

when, $\beta^2 = \dfrac{1}{2} \Rightarrow \omega_2^2 = \dfrac{12E}{\rho L^2} \Rightarrow \omega_2 = 3.46\left[\dfrac{E}{(\rho L^2)}\right]^{\frac{1}{2}}$ rad/s

when, $\beta^2 = 2 \Rightarrow \omega_3^2 = \dfrac{48E}{\rho L^2} \Rightarrow \omega_3 = 6.92\left[\dfrac{E}{(\rho L^2)}\right]^{\frac{1}{2}}$ rad/s

∴ Natural frequencies are, $\omega_1 = 0$

$$\omega_2 = 3.46\left[\frac{E}{(\rho L^2)}\right]^{\frac{1}{2}}$$

$$\omega_3 = 6.96\left[\frac{E}{(\rho L^2)}\right]^{\frac{1}{2}}$$

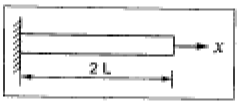
Result: Natural frequencies of longitudinal vibration,

$$\omega_1 = 0$$

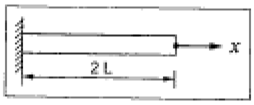
$$\omega_2 = 3.46\left[\frac{E}{(\rho L^2)}\right]^{\frac{1}{2}} \text{ rad/s}$$

$$\omega_3 = 6.96\left[\frac{E}{(\rho L^2)}\right]^{\frac{1}{2}} \text{ rad/s}$$

For the bar as shown in Fig.(i) with length 2 L, modulus of elasticity E, mass density ρ, and cross sectional area A, determine the first two natural frequencies.



Given:



Length, L = 2 L
Young's modulus, E = E
Mass density, ρ = ρ
Cross-sectional area, A = A

To find: Natural frequencies.

Solution: We can divide the bar into two elements as shown in Fig.(iii).

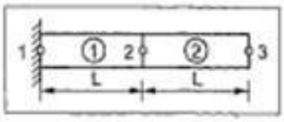


Fig. (iii).

Stiffness matrix for element (1):

$$[K_1] = \frac{AE}{L}\begin{array}{c c} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Similarly,

Element (2):

$$[K_2] = \frac{AE}{L}\begin{array}{c c} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Assembling the element matrix,

$$[K] = \frac{AE}{L}\begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \qquad \dots(1)$$

Lumped mass matrix or consistent mass matrix can be used for solving the problem.

Lumped mass matrix for element (1):

$$[m_1] = \frac{\rho A L}{2} \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \end{array}$$

Similarly,

Element (2):

$$[m_2] = \frac{\rho A L}{2} \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \end{array}$$

Assembling the mass matrix,

$$[m] = \frac{\rho A L}{2} \begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \qquad \ldots (2)$$

Global matrix, for bar element,

$$\{[K] - \omega^2 [m]\}\{u\} = \{P\}$$

$$\left[\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \qquad \ldots (3)$$

Applying the boundary conditions,

$$u_1 = 0 \text{ (fixed)}, \qquad P_1 = 0$$
$$u_2 = u_2 \qquad P_2 = 0$$
$$u_3 = u_3 \qquad P_3 = 0$$

$$\Rightarrow \left[\left[\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\right]\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\left\{\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0 \qquad \ldots (4)$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\left\{\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0 \qquad \ldots (4)$$

To obtain a solution to the set of homogeneous equation in equation (4), we set the determinant of the coefficient matrix equal to zero.

$$\left|\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad \ldots (5)$$

Divide both sides by ρAL,

$$\left| \frac{AE}{\rho A L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \frac{\rho A L}{2 \times \rho A L}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \qquad [\because \lambda = \omega^2]$$

$$\left| \frac{E}{\rho L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Take, $\mu = \frac{E}{\rho L^2}$

$$\left| \mu\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} (2\mu - \lambda) & -\mu \\ -\mu & \left(\mu - \frac{\lambda}{2}\right) \end{vmatrix} = 0 \qquad \ldots (6)$$

$$\Rightarrow \left[(2\mu - \lambda)\left(\mu - \frac{\lambda}{2}\right) \right] - [\mu^2] = 0$$

$$\Rightarrow \left(2\mu^2 - \mu\lambda - \mu\lambda + \frac{\lambda^2}{2} - \mu^2 \right) = 0$$

$$\Rightarrow \mu^2 - 2\mu\lambda + \frac{\lambda^2}{2} = 0$$

$$\Rightarrow \frac{\lambda^2}{2} - 2\mu\lambda + \mu^2 = 0 \qquad \ldots (7)$$

By solving the quadratic equation (7),

$$\lambda = -(-2\mu) \pm \frac{\sqrt{4\mu^2 - \frac{4}{2}\mu^2}}{\left(\frac{2}{2}\right)} \qquad \left[\because \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$\boxed{\lambda = 2\mu \pm \mu\sqrt{2}} \qquad \ldots (8)$$

$$\lambda = \mu[2 \pm \sqrt{2}]$$

$$\therefore \lambda_1 = 3.41\,\mu, \quad \lambda_2 = 0.585\,\mu$$

Natural frequencies are,

We know that, $\lambda = \omega^2$

$$\Rightarrow \omega = \sqrt{\lambda}$$

$$\Rightarrow \omega_1 = \sqrt{3.41\,\mu}$$

$$\boxed{\omega_1 = 1.85\sqrt{\mu} \text{ rad/s}}$$

$$\therefore \omega_1 = 1.85\sqrt{\frac{E}{\rho L^2}} \text{ rad/s} \qquad \left[\because \mu = \frac{E}{\rho L^2}\right]$$

Similarly, $\omega_2 = \sqrt{0.585\,\mu}$

$$\boxed{\omega_2 = 0.76\sqrt{\mu} \text{ rad/sec}}$$

$$\omega_2 = 0.76\sqrt{\frac{E}{\rho L^2}} \text{ rad/s}$$

Result: Natural frequencies are,

$$\omega_1 = 1.85\sqrt{\frac{E}{\rho L^2}} \text{ rad/s}$$

$$\omega_2 = 0.76\sqrt{\frac{E}{\rho L^2}} \text{ rad/s}$$

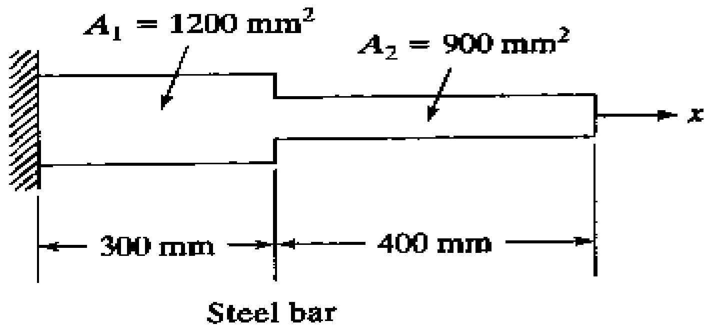
MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject: FINITE ELEMENT METHODS

UNIT – V

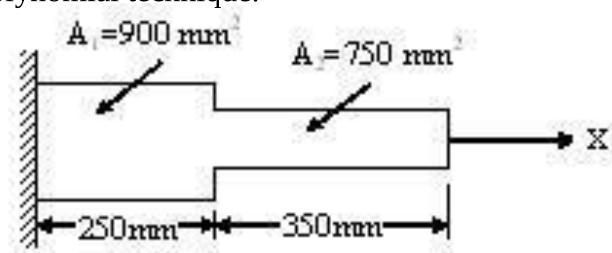
TUTORIAL - V

1. Determine natural frequencies for a Steel bar as shown in figure.

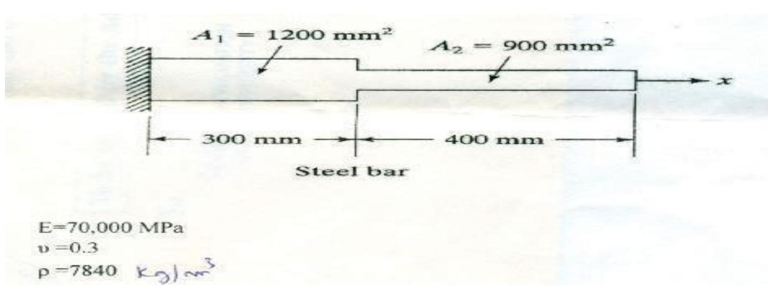


2. a.) Write a short note on damping.

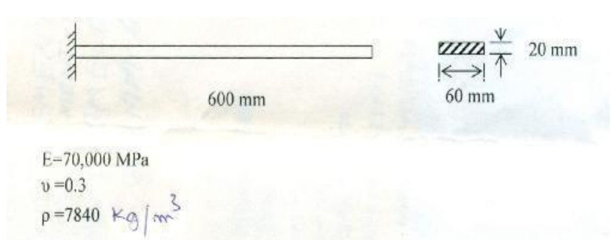
   b.) Consider axial vibration of the steel bar shown in Figure., Develop the global stiffness and mass matrices Determine the natural frequencies and mode shapes using the characteristic polynomial technique.



3. Consider axial vibration of the steel bar shown in Fig. a) Develop the global stiffness and mass matrices b) By hand calculations, determine the lowest natural frequency and mode shape 1and 2



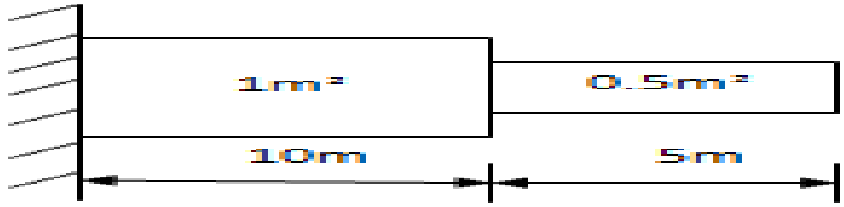
4. Write the step by step procedure to determine the frequencies and nodal displacements of the steel cantilever beam shown in Figure.



5. Explain the Overview of Commercial software's like ANSYS, ABAQUES .

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY
Subject: FINITE ELEMENT METHODS
UNIT – V
ASSIGNMENT - V

1. Determine the Eigen values and Eigen Vectors for the stepped bar as shown in Figure, take density as 7850 kg/m^3 and E= 30X10^6N/m^2?

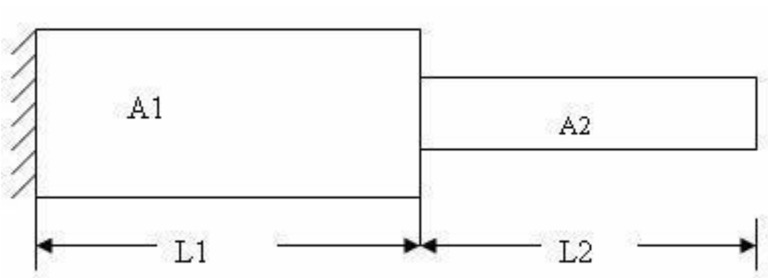


2. Define a.) Eigen value and Eigenvector

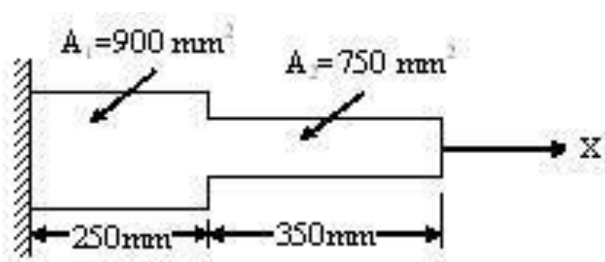
   b.) Dynamic analysis

3. Determine natural frequencies and corresponding mode shapes for the figure

   Take L_1=1m, L_2=2m, A_1=2m^2, A_2=1m^2, ρ = 7850 kg/m^3, E = 200Gpa



4. Consider axial vibration of the steel bar shown in Figure.6,
   i) Develop the global stiffness and mass matrices
   ii) Determine the natural frequencies and mode shapes using the characteristic polynomial technique.



5.) Write short note on a.) Eigen vectors for a stepped beam b.) Evaluation of Eigen values

PREVIOUS QUESTION PAPERS

R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular/supplementary Examinations, April/May 2019

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

***

PART-A (25 Marks)

1). a What is meant by finite Element method **[2M]**
b Name the weighted residual techniques? **[3M]**
c Write down the expression of stiffness matrix for a truss element. **[2M]**
d Define plane strain problem. **[3M]**
e What is CST element? **[2M]**
f Write down the shape functions for an axisymmetric triangular element. **[3M]**
g Write the governing equation for a steady flow heat conduction. **[2M]**
h Write down the expression of stiffness matrix for a beam element. **[3M]**
i What is meant by discretization and assembling? **[2M]**
j What is the difference between static and dynamic analysis? **[3M]**

PART-B (50 MARKS)

SECTION-I

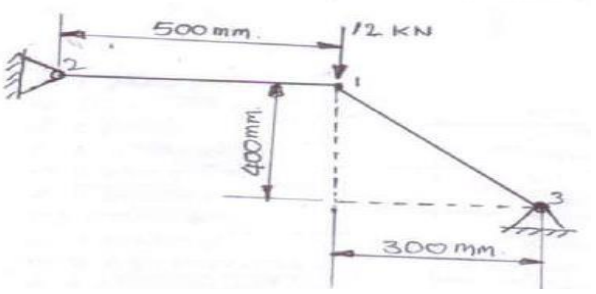
2 Describe advantages, disadvantages and applications of finite element analysis. **[10M]**

OR

3 The following equation is available for a physical phenomena
$\frac{d^2y}{dx^2}$ -10x^2=5; 0<x<1, Boundary Conditions; y(0) =0, y(1) =0, Using Galarkin method of weighted residual find an approximate solution of the above differential equation. **[10M]**

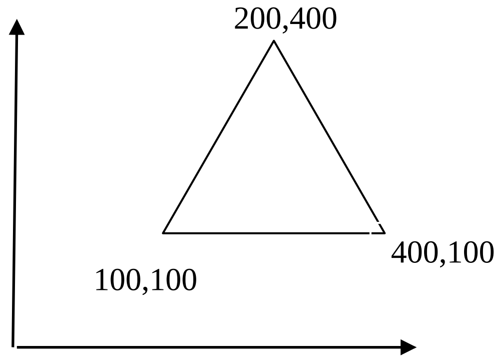
SECTION-II

4 For the two bar truss shown in figure, determine the displacement at node 1 and stresses in element2, Take E=70GPa, A= 200mm^2. **[10M]**



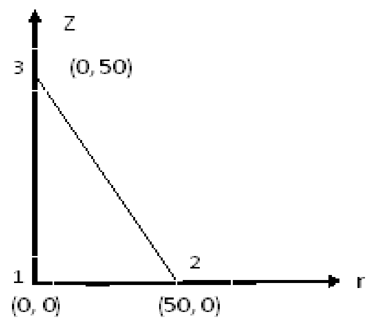
OR

5 For the plane stress element shown in figure the nodal displacements are **[10M]**
U_1= 2.0mm, V_1=1.0mm
U_2= 1.0 mm,V_2= 1.5mm, U_3= 2.5mm,V_3=0.5mm, Take E= 210GPa, ν= 0.25, t=10mm. Determine the strain-Displacement matrix [B].



SECTION-III

6 For axisymmetric element shown in figure, determine the strain-displacement matrix. Let E = $2.1\text{x}10^5\text{N/mm}^2$ and ν= 0.25. The co-ordinates shown in figure are in millimeters. **[10M]**



OR

7 Evaluate the following integral using Gaussian quadrature, so that the result is exact. **[10M]**

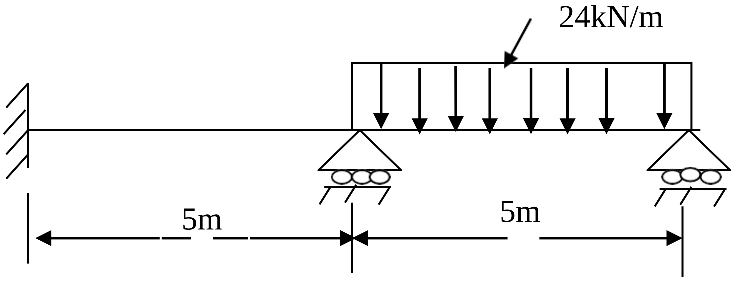
$f(r) = \int_{-1}^{1}(\frac{1}{1+x^2}$ +2x –sinx) dx

SECTION-IV

8 Estimate the temperature distribution in a fin whose cross section is 15mm X 15mm and 500mm long. Take Thermal conductivity as 50W/m-k and convective heat transfer coefficient as 75 W/m^2-k at 25^oC. The base temperature is assumedto be constant and its value may be taken as 900^oC. And also calculate the heat transfer rate? **[10M]**

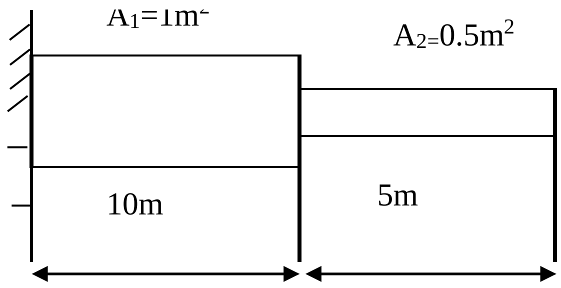
OR

9 For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=$4\text{x}10^6\text{m}^4$. **[10M]**



SECTION-V

10 Determine the Eigen values and Eigen vectors for the beam shown in figure



$E = 30x10^5 N/m^2$
$\rho = 0.283 kg/m^3$

[10M]

OR

11 Write short note on **[10M]**

(a) Eigen vectors for a stepped beam
(b) Evaluation of Eigen values.

****

R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular Examinations, April/May 2018

Finite Element Method

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

******

PART- A

1a. What is the shape function? Give its practical importance. [2]

b) Briefly discuss the Gherkin's approach in solving FEA problems [3]

c) Define is axisymmetric element with 2 practical applications [2]

d. What are the differences between plane stress and plane strain problems [3]

e. Briefly discuss the advantages of Axisymmetric Elements [2]

f. Describe the shape functions in natural coordinates for 2-D Quadrilateral element. [3]

g. Write the governing equation for a steady flow heat conduction [2]

h. Write short notes on applications of FEM [3]

i. What are the practical importance of Eigen values and Eigen vectors [2]

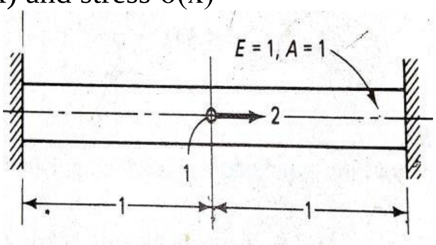
j. Write the Gradient matrix[B] for CST element. [3]

PART – B

10 * 5 = 50 Marks

2. SECTION-1 [5]

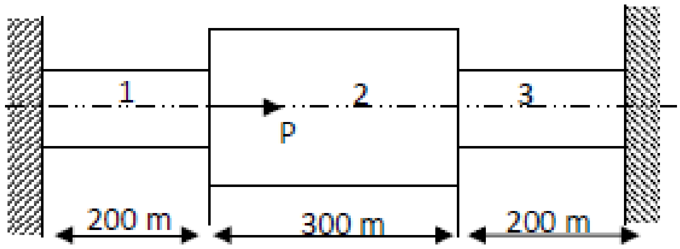
(a) A rod fixed at its ends is subjected to a varying body force as shown in Figure.1. Use the Rayleigh-ritz method with an assumed displacement field $u=a_0+a_1x+a_2x^2$ to determine displacement u(x) and stress σ(x)



(b) Write the Potential function for a continuum under all possible loads and indicate all the variables involved. Also express the total potential of general finite element in terms of nodal displacements [5]

OR

3. An axial load $P = 200X10^3$ N is applied on a bar shown in figure, determine nodal displacements, stress in each material and reaction forces. If $A_1 = 2400$ mm^2, $A_2 = 600$ mm^2, $A_3 = 2000$ mm^2, E_1=70GPa, E_2=200GPa, E_3=67GPa [10]



4. [5]

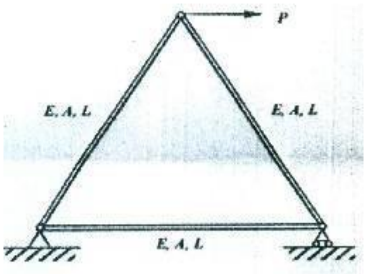
SECTION - II

(a) Derive the B Matrix (relating strains and nodal displacements) for an iso parametric triangular element with linear interpolation for the geometry as well as field variables.

b) Explain why the above element is popularly known as CST. Discuss about the advantages and disadvantages of the element [5]

OR

5. For the truss shown in figure establish the element stiffness matrices and assemble the global stiffness matrix for the active degrees of freedom and determine a) Nodal displacements b) Stress in the members and c) The reaction at the roller support, Take E= 100 Gpa. Area of c/section = 100 mm^2 Length = 100 cm, P = 100 kN. [10]

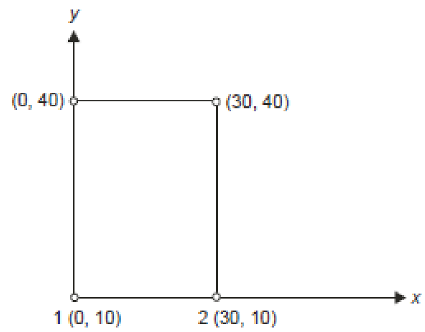


SECTION-III

6. Derive the B Matrix (relating strains and nodal displacements) for an axi-Symmetric iso parametric triangular element with linear interpolation for the geometry as well as field variables. [10]

OR

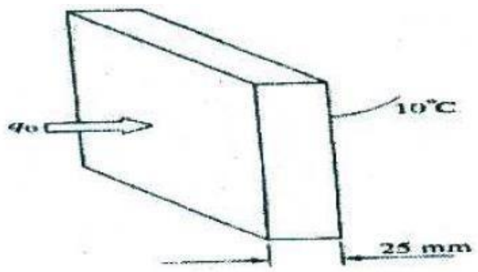
7.(a) Consider a quadrilateral element as shown in figure, Evaluate Jacobian matrix and strain-Displacement matrix at local coordinates ξ =0.5, η = 0.5. [7]



(b) Evaluate the integral $\int_{-1}^{+1} [3e^x + 2x^2 + \frac{1}{(3x+4)}]\, dx$
using one point and two point Gauss quadrature. [3M]

SECTION-IV

8. Heat is entering into a large plate at the rate of q_0=-300w/m2 as shown in Figure, the plate is 25 mm think. The outside surface of the plate is maintained at a temperature of10 ^{0}C. Using two finite elements, solve for the vector of nodal temperatures T, thermal conductivity k=1.0 w/m^{0}c [10]

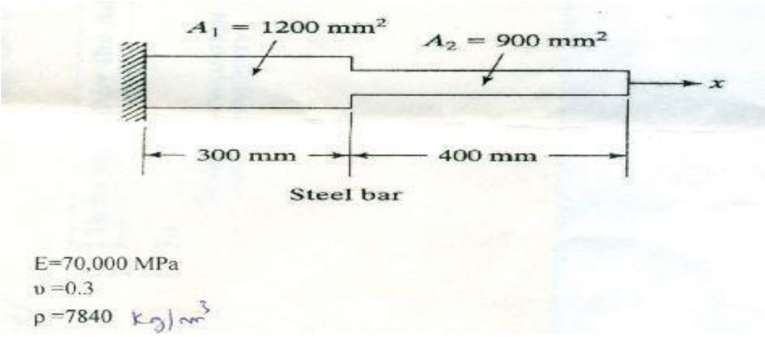


OR

9. Estimate the temperature profile in **a** fin of diameter 25 mm, whose length is 400mm. The thermal conductivity of the fin material is 50 W/m K and heat transfer coefficient over the surface of the fin is 50 W/m^2 K at 30^{0}C. The tip is insulated and the base is exposed to a temperature of 150 ^{0}C. Evaluate the temperatures at points separated by 100 mm each. [10]

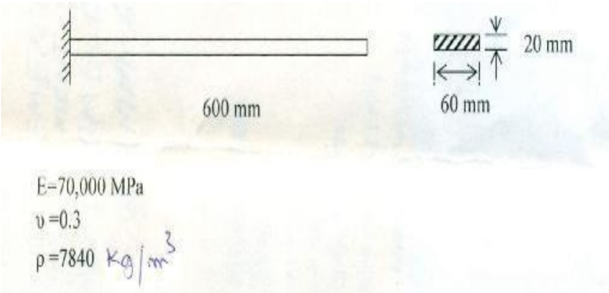
SECTION-V

10. Consider axial vibration of the steel bar shown in Fig. a) Develop the global stiffness and mass matrices b) By hand calculations, determine the lowest natural frequency and mode shape 1and 2 [10]



OR

11. Write the step by step procedure to determine the frequencies and nodal displacements of the steel cantilever beam shown in Fig. [10]



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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester supplementary Examinations, Nov/Dec 2018

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

PART – A

1.a. Briefly discuss weighted residual method for giving approximate solutions for complicated domains [2M]
b. Write the stiffness matrix for 1-d element with linear interpolation functions [3M]
c. Differentiate iso-parametric, sub-parametric, and super parametric elements? [2M]
d. What is the difference between plane truss and space truss? [3M]
e. What are the uses of natural coordinates in 2d- Quadrilateral elements [2M]
f. What are the suitable applications of axi-symmetric elements in FEM? [3M]
g. Write the governing equation for FEA formulation for a fin [2M]
h Express the stiffness matrix for a 1-D conduction problem [3M]
i. What do you understand by mode shapes? [2M]
j. How principle of minimum potential energy is useful in dynamic analysis of systems [3M]

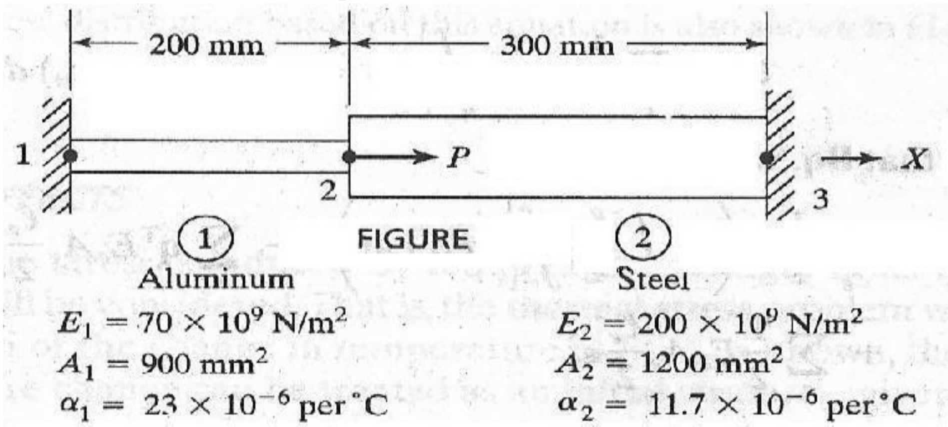
PART – B 10 * 5 = 50 Marks

SECTION-I

2. Derive the equations equilibriums for 3-D body [10M]

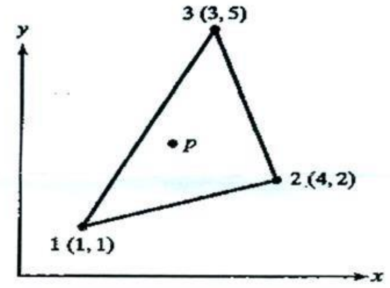
OR

3. An axial load P=300X103N is applied at 200 C to the rod as shown in Figure below. The temperature is the raised to 600 C . [10M]
a) Assemble the K and F matrices.
b) Determine the nodal displacements and stresses.



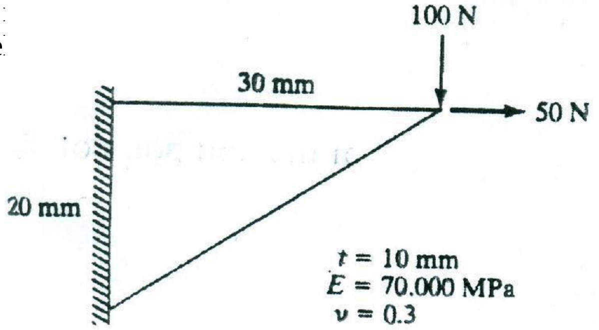
SECTION-II

4. a) Write the difference between CST and LST elements [3M]
b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P. [7M]



OR

5 For the configuration shown in Fig. determine the deflection at the point of load application using a one-element : used, comment on the stress values in the e [10M]

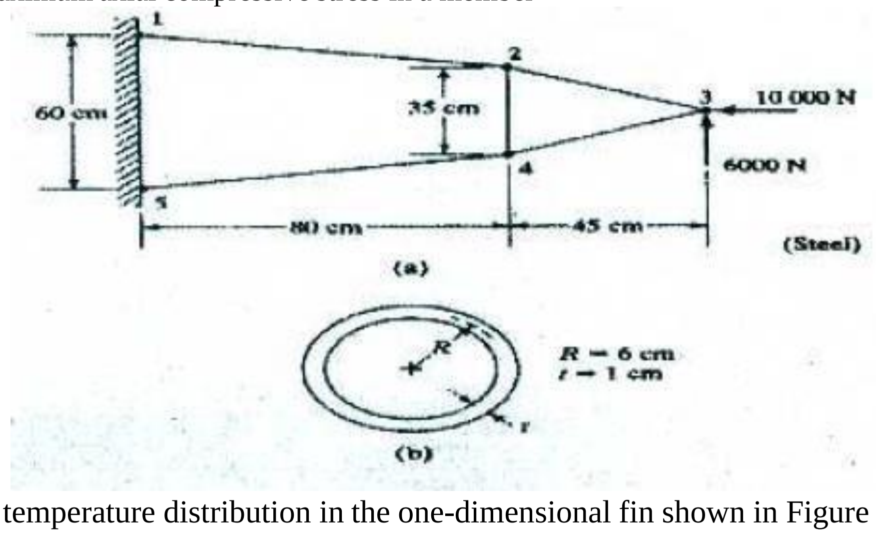


S

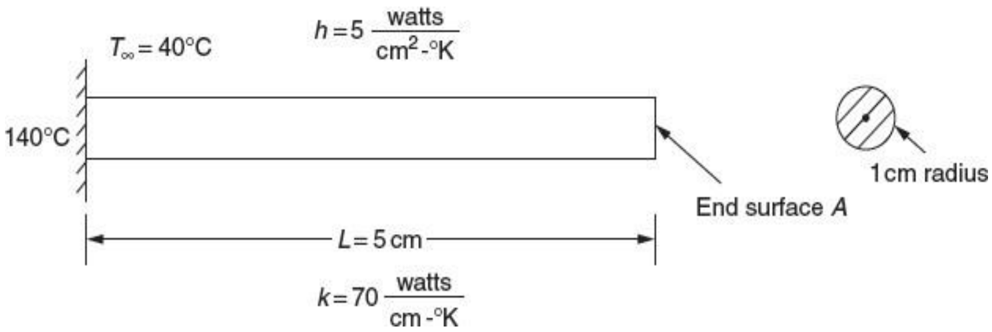
6. Derive the strain displacement matrix for axisymmetric triangular element Discuss advantages of axisymmetric modelling in FEM [10M]

OR

7. Figure shows a five – member steel frame subjected to loads at the free end. The cross section of each member is a tube of wall thickness t=1 cm and mean radius=6cm. Determine the following: [10M]
a) The displacement of node 3 and
b) The maximum axial compressive stress in a member



8. Find the temperature distribution in the one-dimensional fin shown in Figure below using two finite elements. [10M]



OR

9. (a) A 20-cm thick wall of an industrial furnace is constructed using fireclay bricks that have a thermal conductivity of k = 2 W/m-°C. During steady state operation, the furnace wall has a temperature of 800°C on the inside and 300°C on the outside. If oneof the walls of the furnace has a surface area of 2 m^2 (with 20-cm thickness), find the rate of heat transfer and rate of heat loss through the wall. [5M]

(b) A metal pipe of 10-cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20°C while the outer surface of the pipe is at a temperature of 250°C. If the heat transfer coefficient for free convection from the pipe to the air is h = 20 W/m^2-°C find the rate of heat loss from the pipe. [5M]

10. For the two-bar truss shown in Figure below, determine the nodal displacements, element stresses and support reactions. A force of P=1000kN is applied at node-1. Assume E=210GPa and A=600mm2 for each element. [10M]

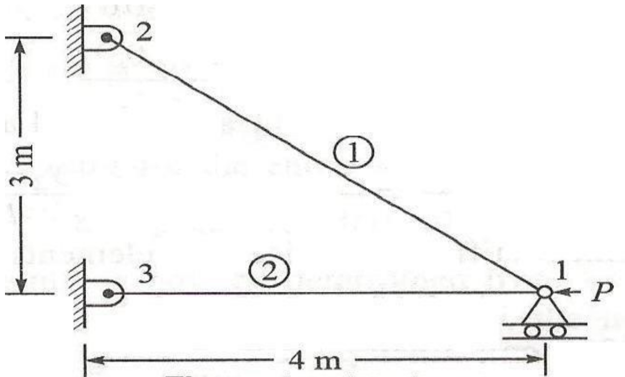


Figure two-bar truss.

OR

11. A bar of length 1 m; cross sectional area 100 mm2; density of 7 gm/cc and Young's modulus 200Gpa is fixed at both the ends. Consider the bar as three bar elements and determine the first two natural frequencies and the corresponding mode shapes. Discuss on the accuracy of the obtained solution [10M]

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MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

B. Tech III Year II Semester

FINITE ELEMENT METHODS

- - -

1.a) Derive the interpolation functions at all nodes for the quadratic serendipity element.

b) Evaluate the integral by using one and two-point Gaussian quadrature and compare with exact value.

$$I = \int_{-1}^{+1}\int_{-1}^{+1} (x^3 + x^2 y + xy^2 + \sin 2x + \cos 2y)\,dx\,dy$$

2.a) Clearly explain the finite element formulation for an axisymmetric shell with an axisymmetric loading. Determine the matrix relating strains and nodal displacements for an axisymmetric triangular element.

b) Establish the Hermite shape functions for a beam element Derive the equivalent nodal point loads for a u.d.l. acting on the beam element in the transverse direction and also determine stiffness matrix.

3.a) Write about different boundary considerations in beams.

b) Determine the support reactions and maximum vertical deflection for the continuous beam shown in Figure.1.

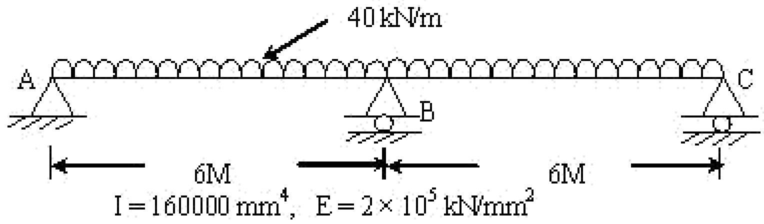

Figure.1

4.a) Discuss in detail about 2D heat conduction in Composite slabs using FEA.

b) Using the isoparametric element, find the Jacobian and inverse of Jacobian matrix for the element shown in Fig.2, 3(a) & 3(b) for the following cases.

i) Determine the coordinate of a point P in x-y coordinate system for the ξ = 0.4 and η = 0.6.

ii) Determine the coordinate of the Q in ξ and η system for the x = 2.5 and y = 1.0.

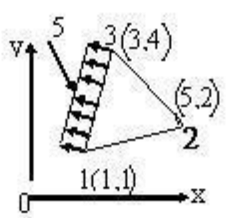

Fig. 2

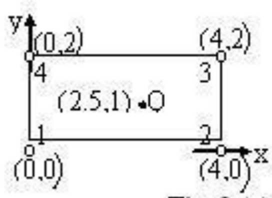

Fig. 3 (a)

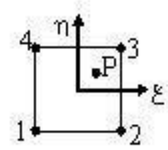

Fig. 3 (b)

5. Calculate the temperature distribution and the heat dissipating capacity of a fin shown in Figre.4. The thermal conductivity of the material is 200 W/ m K. The surface transfer coefficient is 0.5 W/m^{2}K. The ambient temperature is 30°C. the thickness of the fin is 1 cm.

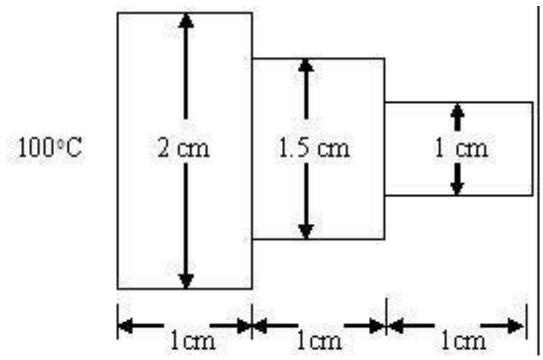

Figure.4

6.a) Write the steps involved in finite-element analysis of a typical problem.

b) Determine the nodal displacements, element stresses and support reactions for the bar as shown in Figure 5. Take E = 200 × 10^9 N/m^2.

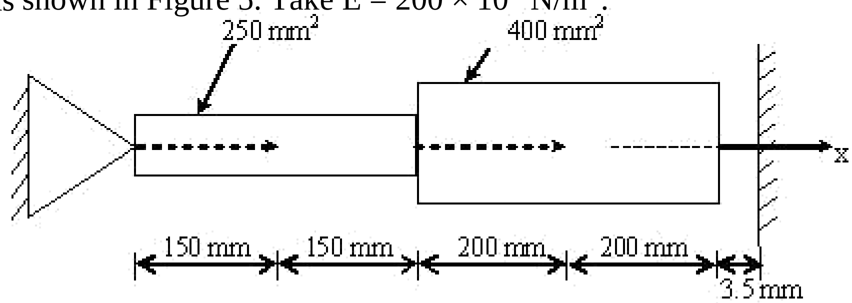

Figure.5

7.a) Derive the equilibrium equation for an elastic continuum using potential energy by displacement approach.

b) Explain the following methods used for the formulation of element characteristics and load matrices:

i) Variational approach   ii) Galerkin approach

8.a) With an example differentiate Between Lumped mass, Consistent mass and Hybrid mass matrix and derive for truss element.

b) Consider axial vibration of the steel bar shown in Figure.6,

i) Develop the global stiffness and mass matrices

ii) Determine the natural frequencies and mode shapes using the characteristic polynomial technique.

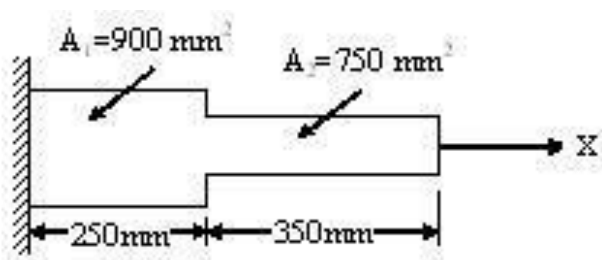

Figure.6

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

B. Tech III Year II Semester
FINITE ELEMENT METHODS
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1)a) Discuss in detail about the concepts of FEM formulation .How is that FEM emerged as powerful tool. Discuss in detail about applications of finite element method.

b)Derive an equation for finding out the potential energy by Rayleigh –Ritz method. Using Rayleigh – Ritz method, find the displacement of the midpoint of the rod shown in Fig.1. Assume E = 1, A = 1, ρ g = 1 by using linear and quadratic shape functions concept.

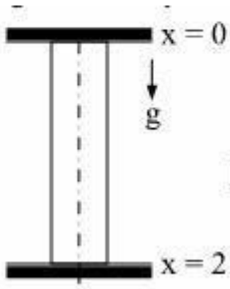

Fig. 1

2. a) Discuss in detail about Linear and Quadratic shape functions with examples.
   b) For the truss shown in fig.2 determine the displacements at point B and stresses in the bars by considering linear and quadratic shape functions.

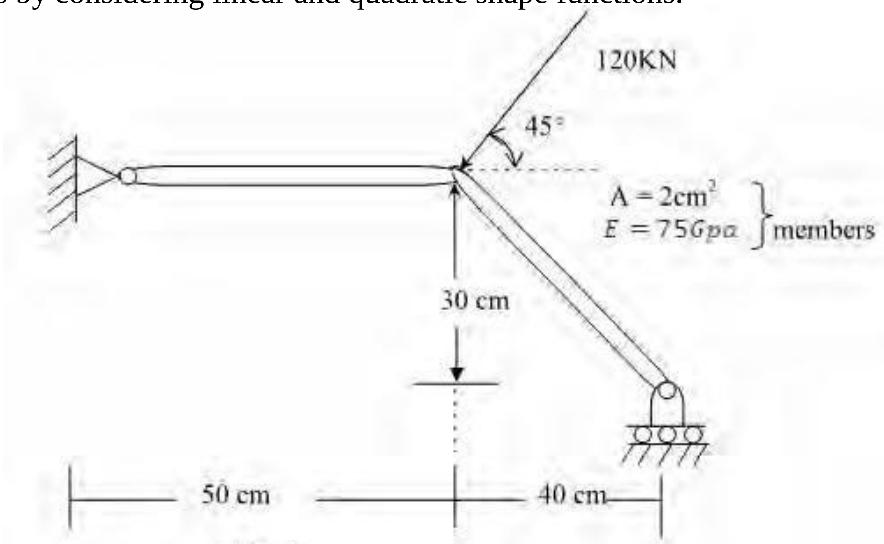

Fig. 2

3. a) Consider axial vibration of the Aluminum bar shown in Fig.3, (i) develop the global stiffness and (ii) determine the nodal displacements and stresses using elimination approach and with help of linear and quadratic shape function concept. Assume Young's Modulus E = 70Gpa.

   b) Determine the mass matrix for truss element with an example.

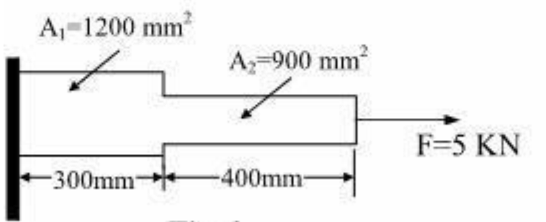

Fig. 3

4. a) Establish the shape functions for a 3 – noded triangular element.

   b) Find the deformed configuration, and the maximum stress and minimum stress locations for the rectangular plate loaded as shown in the fig.4. Solve the problem using 2 triangular elements. Assume thickness = 10cm; E = 70 Gpa, and υ = 0.33.

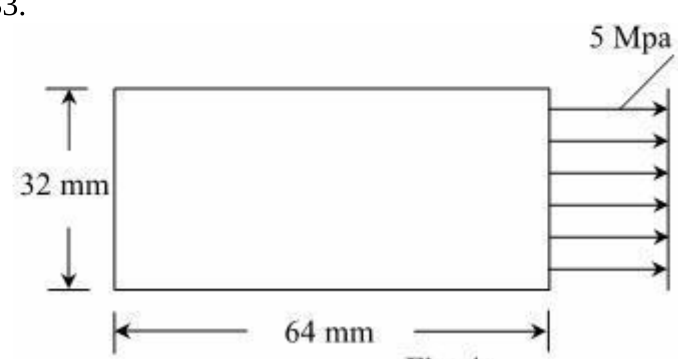

Fig. 4

5. a) Determine the shape functions for 4 – nodded quadrilateral element.

   b) For a beam and loading shown in fig.5, determine the slopes at 2 and 3 and the vertical deflection at the midpoint of the distributed load.

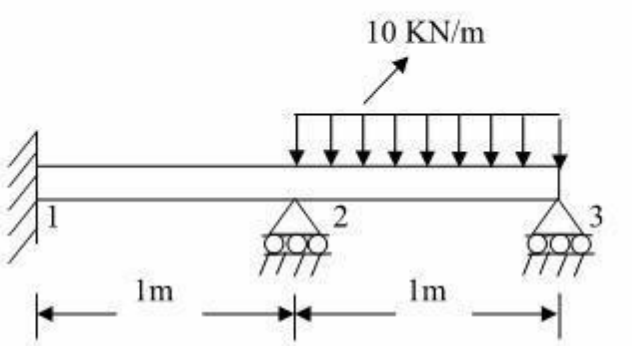

Fig.5

6. a) Clearly explain the finite element formulation for an axisymmetric shell with an axisymmetric loading.Determine the matrix relating strains and nodal displacements for an axisymmetric triangular element.

   b) Determine the temperature distribution in a straight fin of circular c/s. Use three one dimensional linear elements and consider the tip is insulated. Diameter of fin is 1 cm, length is 6 cm, h = 0.6 W/cm^2 –C, φ_∞ = 25^{0}C and base temperature is φ_1 =80^{0}C.

7. a) Determine the element stresses, strains and support reactions for the given bar problem as shown in Fig. 6

$=1.2\ mm$; $L = 150\ mm$; $P = 60000\ N$; $E = 2\ X\ 10^4\ \frac{N}{m}$; $A = 250\ mm$.

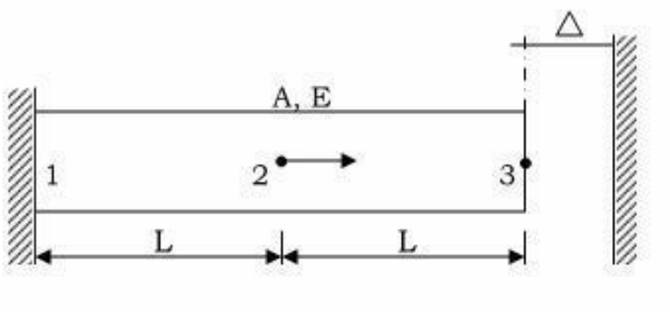

Fig. 6

   b) What are shape functions? Indicate briefly the role of shape functions in FEM analysis.

8. a) Derive one dimensional steady state heat conduction equation.

   b) An axisymmetric triangular element is subjected to the loading as shown in fig.7 the load is distributed throughout the circumference and normal to the boundary. Derive all the necessary equations and derive the nodal point loads.

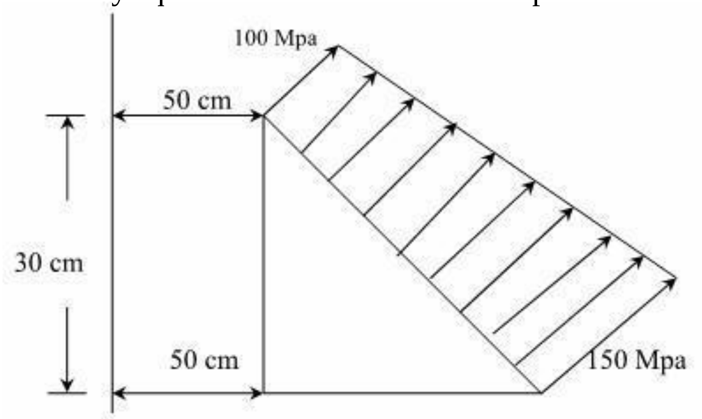

Fig.7

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Finite Element Methods

1.a) Write the strain stress relations based on generalized Hooke's law and derive the elasticity matrix for 3-D field problems.

c) Describe the standard procedure to be followed for understanding the finite element method step by step with suitable example.

2.a) Derive the stiffness matrix of axial bar element with quadratic shape functions based on first principles.

c) Calculate the nodal displacements and forces for the stepped bar with the stiffness values of 10 kN/m and 18 kN/m and a load of 32 kN is subjected at the end of the stepped bar and other end of the bar is fixed.

3.a) Derive the shape functions and stiffness matrix of a two nodded beam element.

c) Derive the load vector for the beam element when a uniformly distributed load is applied.

4.a) For a plane strain problem, the nodal displacements are u_1 = 4.4 μm, u_2 = 2.2 μm, u_3=2.2 μm, v_1 = 3.8 μm, v_2 = 2.9 μm, v_3 = 4.5 μm. Take E=200 GPa, μ = 0.3 and t=10mm. Find the stresses, principal stresses. The coordinates of triangular element are 1(5,25), 2(15,5) and 3(25,15). All dimensions are in millimeters.

c) Show that the stiffness for a triangular element is $[B]^T[D][B]$ At using variational principle. Where A=area of the triangle and t= thickness.

5.a) Compute the strain displacement matrix and also the strains of a axisymmetric triangular element with the coordinates r_1 = 3 cm, z_1 = 4 cm, r_2 = 6 cm, z_2 = 5 cm, r_3= 5 cm, z_3 = 8 cm. The nodal displacement values are u_1 = 0.01 mm, w_1 = 0.01 mm, u_2 = 0.01 mm, w_2 = -0.04 mm, u_3 = -0.03 mm, w_3 = 0.07 mm

b) Differentiate between Axi symmetric elements and symmetric elements with suitable examples.

6.a) Explain the methodology to estimate the stiffness matrix of four noded quadrilateral element.

b) Evaluate $\int [e^{2x} + x^3 + 1 / (x^2 + 2)]\, dx$ over the limits -1 and +1 using one point and three point quadrature formula and compare with exact solution.

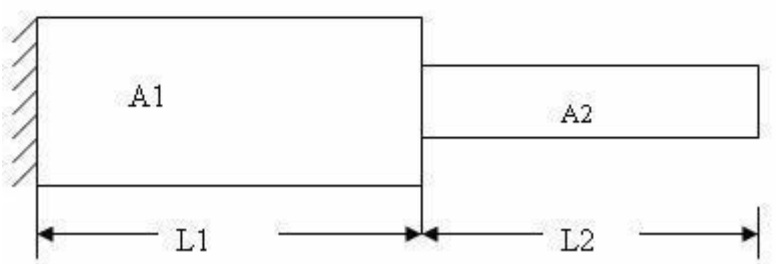
7.a) What are different thermal applications of finite element analysis? Compare the structural analysis with thermal analysis.

b) Calculate the temperature distribution in the fin of 10 mm diameter, which is exposed to the convective b.c. of 40 W/m^2 K with 30^0 C. The base of the fin is exposed to a heat flux of 450 kW/m^3 and the thermal conductivity of fin material is 30 W/m K.

8. Determine natural frequencies and corresponding mode shapes for the figure 8.

Take L_1=1m, L_2=2m, A_1=2m^2, A_2=1m^2, ρ = 7850 kg/m^3, E = 200Gpa

Fig: 8



R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, October/November 2020

Finite Element Method

(ME)

Roll No										

Time: 2 hours **Max. Marks: 75**

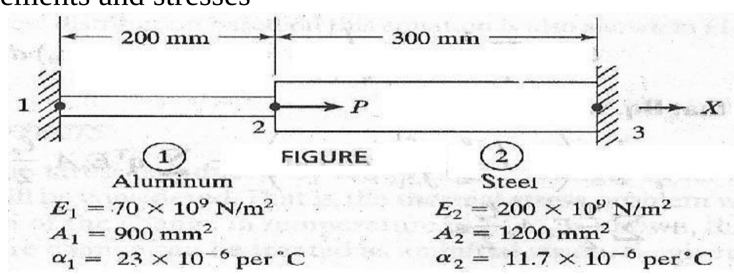
Answer Any **Four** Questions

All Questions carries equal marks.

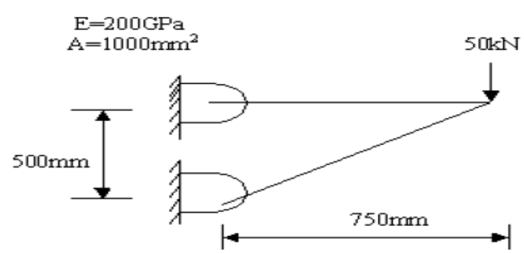
***

1 Briefly describe the general procedure of finite element analysis.

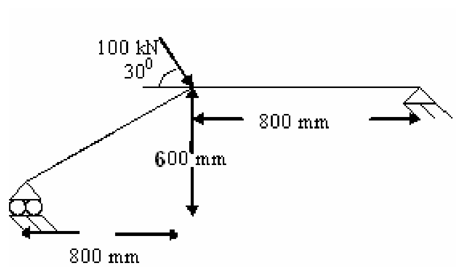
2 An axial load P =300X10^{3}N is applied at 20^0 C to the rod as shown in Figure below. The temperature is the raised to 60^0 C a) Assemble the K and F matrices b) Determine the nodal displacements and stresses



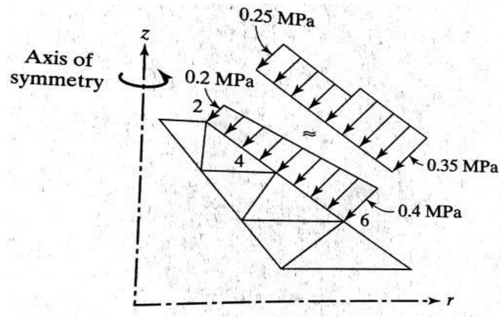
3 Determine the stiffness matrix, stresses and reactions in the truss structure shown in Figure.



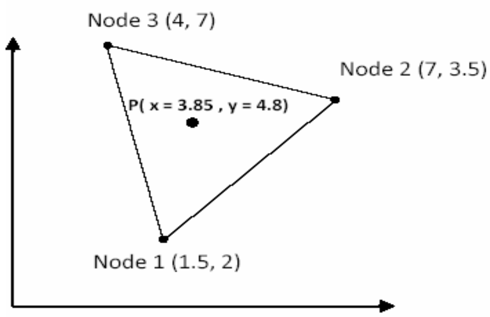
4 Estimate the displacement vector, strains, stresses and reactions in the truss structure shown below in figure. Take A = 1000 mm^2 and E = 200 GPa



5 An axisymmetric body with a linearly distributed load on the conical surface is shown in Fig. Determine the equivalent point loads at nodes 2, 4 and 6.



6 An Isoparametric constant strain triangular element is shown in Figure.
i) Evaluate the shape functions N1, N2 and N3 at an intermediate point P for the triangular element.
ii) Determine the Jacobean of transformation J for the element.



7 Describe heat transfer analysis for straight fin

8 Obtain the Eigen values and Eigen vectors for the cantilever beam of length 2 m using consistent mass for translation DOF with E = 200GPa, ρ =7500kg/m^3.

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Code No: R17A0320 **R17**

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular Examinations, October/November 2020

Finite Element Methods

(ME)

Roll No										

Time: 2 hours **Max. Marks: 70**

Answer Any **Four** Questions
All Questions carries equal marks.
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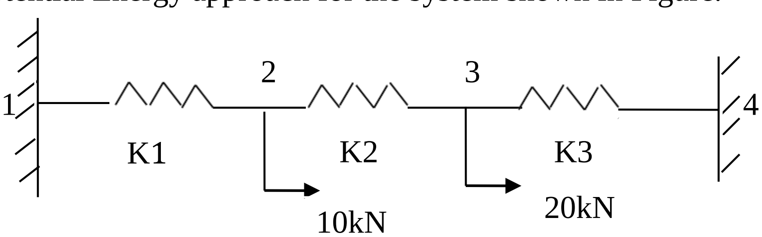
1 a Discuss how finite element method is evolved in the engineering field.

b Discuss the advantages and disadvantages of Finite Element Method

2 Solve for the nodal displacement and support reactions, using the principle of Min. Potential Energy approach for the system shown in Figure.



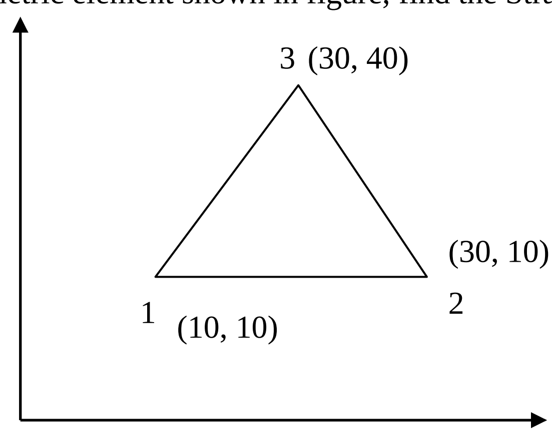
K1=1200kN/m
K2=1800kN/m
K3=1500kN/m

3 Derive stiffness matrix for a Truss bar Element

4 Derive the stiffness matrix for a Three noded CST Element.

5 a What is an axi-symmetric problem?

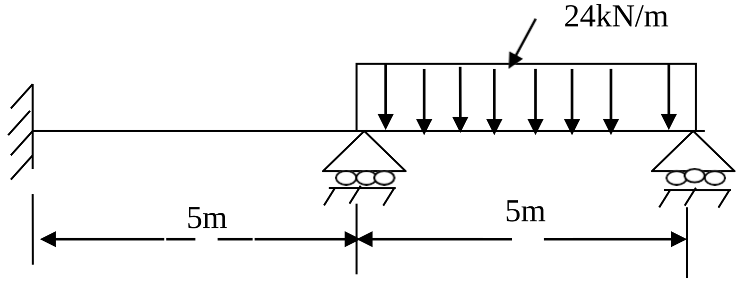
b For the Axi-symmetric element shown in figure, find the Strain-Displacement Matrix.



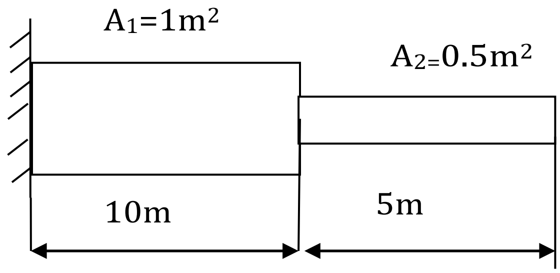
6 Use Gaussian Quadrature to obtain the exact value of the integral

$$f(x) = \int_{-1}^{1} \frac{1}{1+x^2} + 2x - \sin x$$

7 For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=$4\text{x}10^6\text{m}^4$.



8 Determine the Eigen values and Eigen vectors for the beam shown in figure



E=$30\text{x}10^5\text{N/m}^2$

P=0.283kg/m^3

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Code No: **R17A0320** **R17**

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, February 2021

Finite Element Method

(ME)

Roll No										

Time: 2 hours 30 min **Max. Marks: 70**

Answer Any **Five** Questions

All Questions carries equal marks.

***

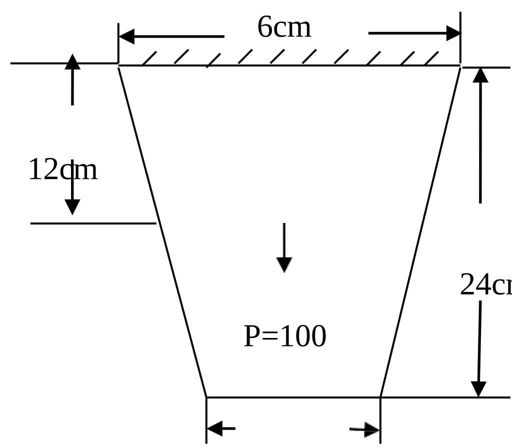
1 Derive the equations of equilibrium of a 3-Dimensional stressed body. **[14M]**

2 Consider the thin (steel) plate shown in figure. The plate has a uniform thickness t=10mm, Young's modulus $E=20 \times 10^9 N/m^2$. **[14M]**

a) Using the elimination approach, solve for the global displacement vector

b) Evaluate the stresses in each element.

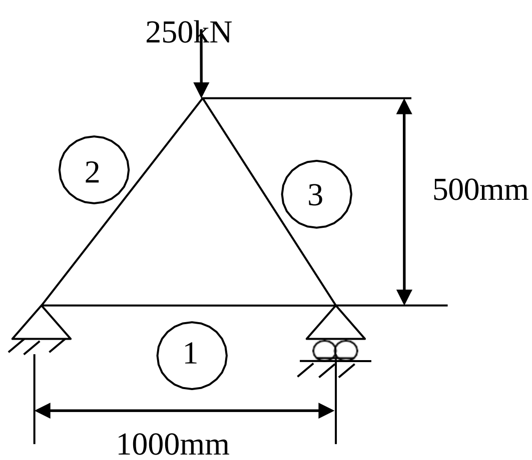
c) Determine the reaction force at the support.



3 Consider a three bar truss as shown in figure. It is given that $E=2X10^5 N/mm^2$. Calculate the following: **[14M]**

(i) Nodal displacements (ii)Stress in each member

(iii) Reactions at the support.

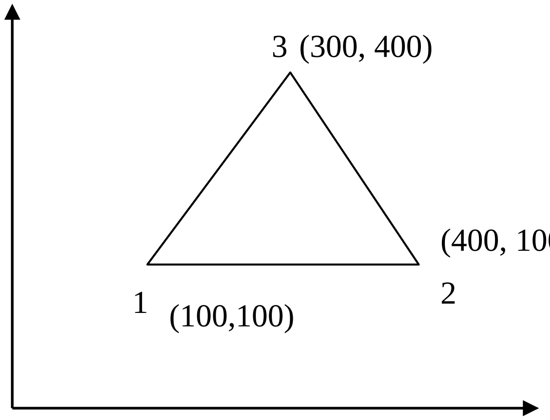


Take:
A_1=2000 mm^2
A_2= 2500 mm^2
A_3= 2500 mm^2

4 a What are the elements commonly used in the analysis of 2-Dimensional roblem? **[04M]**

	b	Derive Strain-Displacement matrix for the 3-noded triangular element.	**[10M]**
5		For the plane stress element shown in figure the nodal displacements are	**[14M]**

U_1= 2 mm $\quad$ V_1= 1 mm
U_2= 1mm $\quad$ V_2= 1.5mm
U_3 = 2.5mm $\quad$ V_3= 0.5mm



Determine the element stresses. Assume E=200GN/m^2, υ = 0.3, t= 10mm.

6 Use Gaussian Quadrature to obtain the exact value for the following integral. **[14M]**

$$\int_{-1}^{1}(r^3 - 1)(s^2 + s)dr.ds$$

7 A wall consists of 4cm thick wood, 10cm thick glass fiber insulation and 1cm thick plaster. If the temperature on the wood and plaster faces are 20°C and -20°C respectively. Determine the temperature distribution in the wall with 1D linear element approach. Assume thermal conductivity of wood, glass and plaster as 0.17, 0.035 and 0.5 W/m-°C. The convective heat transfer coefficient on the colder side of the wall as 25W/m^2-°C. **[14M]**

8 Write short note on

(a) Formulation of Finite Element model in dynamic analysis **[4M]**
(b) Eigen vectors for a stepped bar. **[10M]**

****

R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, February 2021

Finite Element Method

(ME)

Roll No										

Time: 2 hours 30 min **Max. Marks: 75**

Answer Any **Five** Questions

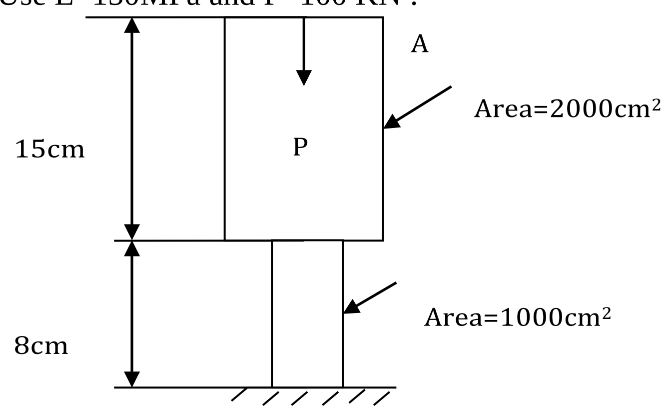
All Questions carries equal marks.

***

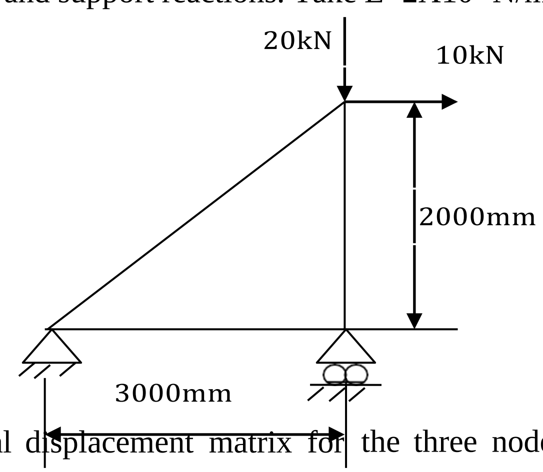
1 a Enumerate thegeneralized procedure involved in Finite Element Method **[10M]**

b Discuss the different engineering applications of Finite Element Method **[05M]**

2 For the vertical bar shown in figure, find the deflection at 'A' and the stress distribution. Use E=150MPa and P=100 KN . **[15M]**



3 Consider the plane truss shown in figure, determine the nodal displacements, Element forces and support reactions. Take $E=2X10^5$ N/mm^2; $A= 1500mm^2$. **[15M]**



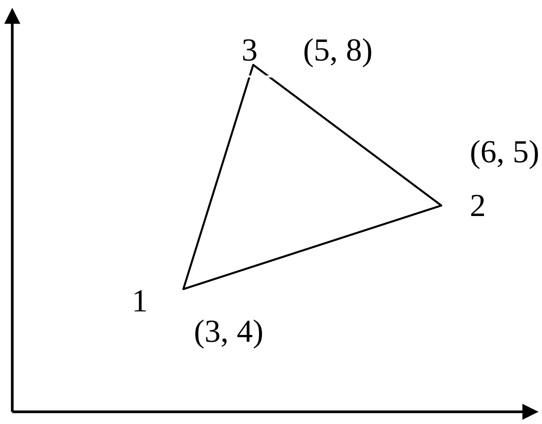
4 Compute Nodal displacement matrix for the three noded triangular element **[15M]**

shown in figure and also determine the element strains, if the nodal displacements are given as

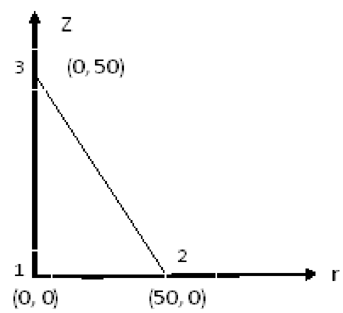
U_1= 0.002 cm V_1= 0.001cm E=200 Gpa & υ = 0.25

U_2= 0.001 cm V_2= -0.004 cm

$U_3 = -0.003$ cm $\qquad V_3 = 0.007$ cm



5		For axi-symmetric element shown in figure, determine the stiffness matrix. Let $E = 2.1 \times 10^5 N/mm^2$ and $\upsilon = 0.25$. The co-ordinates shown in figure are in millimeters.	**[15M]**



6		Derive the stiffness matrix for a four noded isoparametric quadrilateral element.	**[15M]**
7		Estimate the temperature distribution in a fin whose cross section is 10mmx10mm and 500mm long. Take thermal conductivity as 50W/m-k and convective heat transfer coefficient as $75W/m^2k$ at 25°C. The base temperature is assumed to be constant and its value may be taken as 900°C. And also calculate heat transfer rate?	**[15M]**
8	a	Distinguish between lumped mass and consistent mass matrices	**[06M]**
	b	Derive the consistent mass matrix for an one dimensional bar element.	**[09M]**

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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular/supplementary Examinations, April/May 2019

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B

Part A is compulsory which carriers 25 marks and Answer all questions.

Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

***

PART-A (25 Marks)

1). a What is meant by finite Element method **[2M]**

b Name the weighted residual techniques? **[3M]**

c Write down the expression of stiffness matrix for a truss element. **[2M]**

d Define plane strain problem. **[3M]**

e What is CST element? **[2M]**

f Write down the shape functions for an axisymmetric triangular element. **[3M]**

g Write the governing equation for a steady flow heat conduction. **[2M]**

h Write down the expression of stiffness matrix for a beam element. **[3M]**

i What is meant by discretization and assembling? **[2M]**

j What is the difference between static and dynamic analysis? **[3M]**

PART-B (50 MARKS)

SECTION-I

2 Describe advantages, disadvantages and applications of finite element analysis. **[10M]**

OR

3 The following equation is available for a physical phenomena
$\frac{d^2 y}{dx^2}$ -10x^2=5; 0<x<1, Boundary Conditions; y(0) =0, y(1) =0, Using Galarkin method of weighted residual find an approximate solution of the above differential equation. **[10M]**

SECTION-II

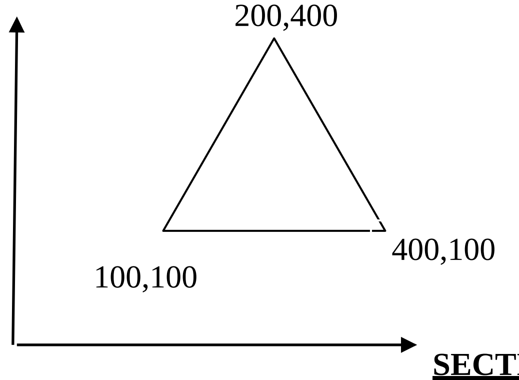
4 For the two bar truss shown in figure, determine the displacement at node 1 and stresses in element2, Take E=70GPa, A= 200mm^2. **[10M]**

OR

5 For the plane stress element shown in figure the nodal displacements are
U_1= 2.0mm, V_1=1.0mm

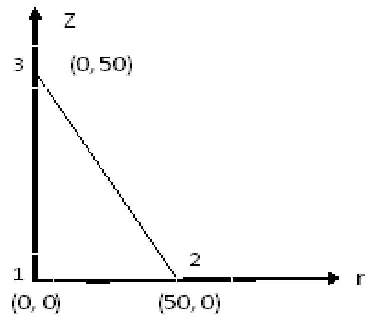
U_2 = 1.

0 mm, V_2= 1.5mm, U_3= 2.5mm, V_3=0.5mm, Take E= 210GPa, ν= 0.25, t=10mm. Determine the strain-Displacement matrix [B]. **[10M]**



SECTION-III

6 For axisymmetric element shown in figure, determine the strain-displacement matrix. Let E = $2.1x10^5$N/mm^2 and ν= 0.25. The co-ordinates shown in figure are in millimeters. **[10M]**



OR

7 Evaluate the following integral using Gaussian quadrature, so that the result is exact. **[10M]**

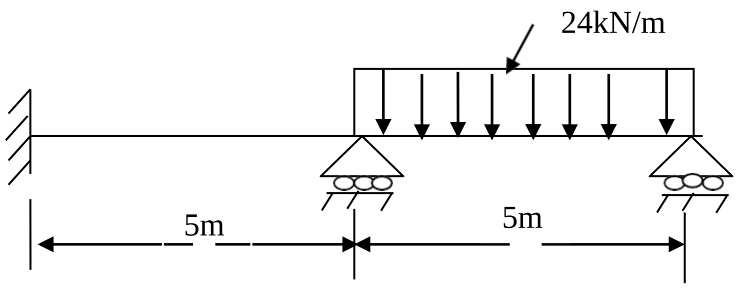
$f(r) = \int_{-1}^{1}(\frac{1}{1+x^2}$ +2x –sinx) dx

SECTION-IV

8 Estimate the temperature distribution in a fin whose cross section is 15mm X 15mm and 500mm long. Take Thermal conductivity as 50W/m-k and convective heat transfer coefficient as 75 W/m^2-k at 25^{o}C. The base temperature is assumedto be constant and its value may be taken as 900^{o}C. And also calculate the heat transfer rate? **[10M]**

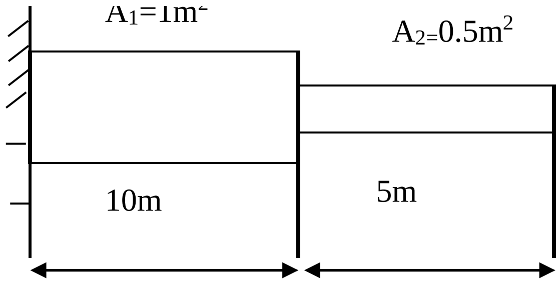
OR

9 For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=$4x10^6$m^4. **[10M]**



SECTION-V

10 Determine the Eigen values and Eigen vectors for the beam shown in figure



$E=30x10^5N/m^2$
$\rho=0.283kg/m^3$ **[10M]**

OR

11 Write short note on **[10M]**

(a) Eigen vectors for a stepped beam
(b) Evaluation of Eigen values.

****

R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular Examinations, April/May 2018

Finite Element Method

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

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PART- A

1a. What is the shape function? Give its practical importance. [2]

b) Briefly discuss the Gherkin's approach in solving FEA problems [3]

c) Define is axisymmetric element with 2 practical applications [2]

d. What are the differences between plane stress and plane strain problems [3]

e. Briefly discuss the advantages of Axisymmetric Elements [2]

f. Describe the shape functions in natural coordinates for 2-D Quadrilateral element. [3]

g. Write the governing equation for a steady flow heat conduction [2]

h. Write short notes on applications of FEM [3]

i. What are the practical importance of Eigen values and Eigen vectors [2]

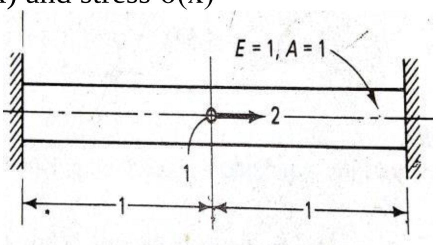
j. Write the Gradient matrix[B] for CST element. [3]

PART – B

10 * 5 = 50 Marks

2. SECTION-1 [5]

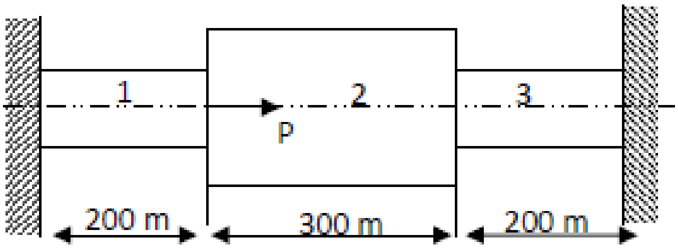
(a) A rod fixed at its ends is subjected to a varying body force as shown in Figure.1. Use the Rayleigh-ritz method with an assumed displacement field $u=a_0+a_1x+a_2x^2$ to determine displacement $u(x)$ and stress $\sigma(x)$



(b) Write the Potential function for a continuum under all possible loads and indicate all the variables involved. Also express the total potential of general finite element in terms of nodal displacements [5]

OR

3. An axial load $P = 200X10^3$ N is applied on a bar shown in figure, determine nodal displacements, stress in each material and reaction forces. If A_1= 2400 mm^2, A_2= 600 mm^2, A_3= 2000 mm^2, E_1=70GPa, E_2=200GPa, E_3=67GPa [10]



4. [5]

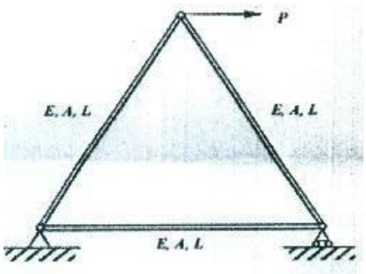
SECTION - II

(a) Derive the B Matrix (relating strains and nodal displacements) for an iso parametric triangular element with linear interpolation for the geometry as well as field variables.

b) Explain why the above element is popularly known as CST. Discuss about the advantages and disadvantages of the element [5]

OR

5. For the truss shown in figure establish the element stiffness matrices and assemble the global stiffness matrix for the active degrees of freedom and determine a) Nodal displacements b) Stress in the members and c) The reaction at the roller support, Take E= 100 Gpa. Area of c/section = 100 mm^2 Length = 100 cm, P = 100 kN. [10]

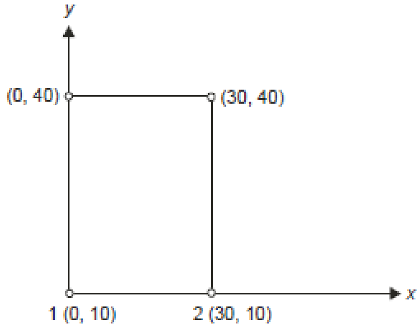


SECTION-III

6. Derive the B Matrix (relating strains and nodal displacements) for an axi-Symmetric iso parametric triangular element with linear interpolation for the geometry as well as field variables. [10]

OR

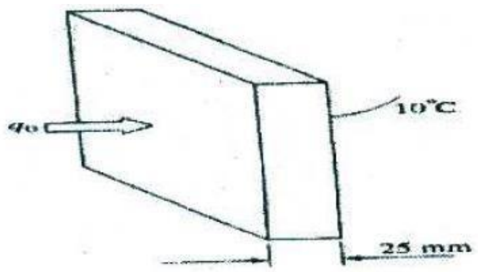
7.(a) Consider a quadrilateral element as shown in figure, Evaluate Jacobian matrix and strain-Displacement matrix at local coordinates ξ =0.5, η = 0.5. [7]



(b) Evaluate the integral $\int_{-1}^{+1} [3e^x + 2x^2 + \frac{1}{(3x+4)}]\, dx$ using one point and two point Gauss quadrature. [3M]

SECTION-IV

8. Heat is entering into a large plate at the rate of q_0=-300w/m2 as shown in Figure, the plate is 25 mm think. The outside surface of the plate is maintained at a temperature of10 ^{0}C. Using two finite elements, solve for the vector of nodal temperatures T, thermal conductivity k=1.0 w/m^{0}c [10]

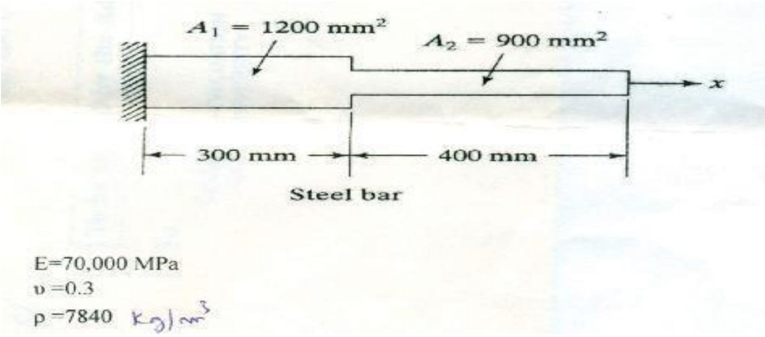


OR

9. Estimate the temperature profile in **a** fin of diameter 25 mm, whose length is 400mm. The thermal conductivity of the fin material is 50 W/m K and heat transfer coefficient over the surface of the fin is 50 W/m^2 K at 30^{0}C. The tip is insulated and the base is exposed to a temperature of 150 ^{0}C. Evaluate the temperatures at points separated by 100 mm each. [10]

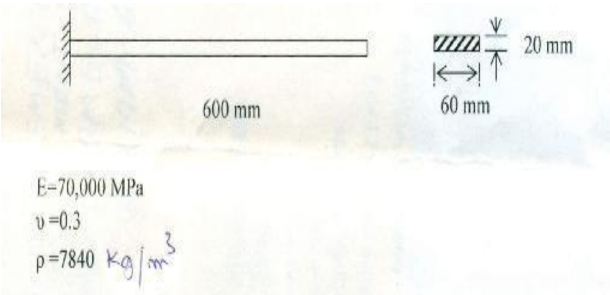
SECTION-V

10. Consider axial vibration of the steel bar shown in Fig. a) Develop the global stiffness and mass matrices b) By hand calculations, determine the lowest natural frequency and mode shape 1and 2 [10]



OR

11. Write the step by step procedure to determine the frequencies and nodal displacements of the steel cantilever beam shown in Fig. [10]



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Code No: R15A0322

R15

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester supplementary Examinations, Nov/Dec 2018

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B

Part A is compulsory which carriers 25 marks and Answer all questions.

Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

PART – A

1.a. Briefly discuss weighted residual method for giving approximate solutions for complicated domains [2M]

b. Write the stiffness matrix for 1-d element with linear interpolation functions [3M]

c. Differentiate iso-parametric, sub-parametric, and super parametric elements? [2M]

d. What is the difference between plane truss and space truss? [3M]

e. What are the uses of natural coordinates in 2d- Quadrilateral elements [2M]

f. What are the suitable applications of axi-symmetric elements in FEM? [3M]

g. Write the governing equation for FEA formulation for a fin [2M]

h Express the stiffness matrix for a 1-D conduction problem [3M]

i. What do you understand by mode shapes? [2M]

j. How principle of minimum potential energy is useful in dynamic analysis of systems [3M]

PART – B 10 * 5 = 50 Marks

SECTION-I

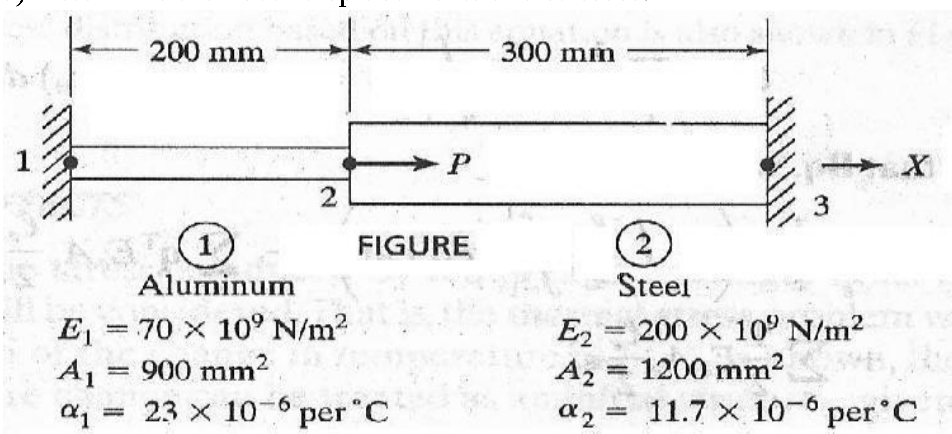
2. Derive the equations equilibriums for 3-D body [10M]

OR

3. An axial load P=300X103N is applied at 200 C to the rod as shown in Figure below. The temperature is the raised to 600 C . [10M]

a) Assemble the K and F matrices.

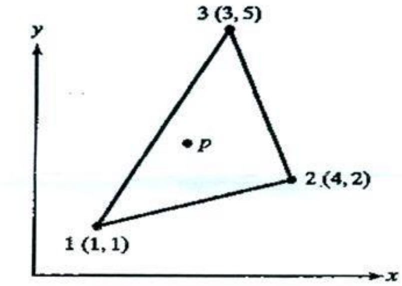
b) Determine the nodal displacements and stresses.



SECTION-II

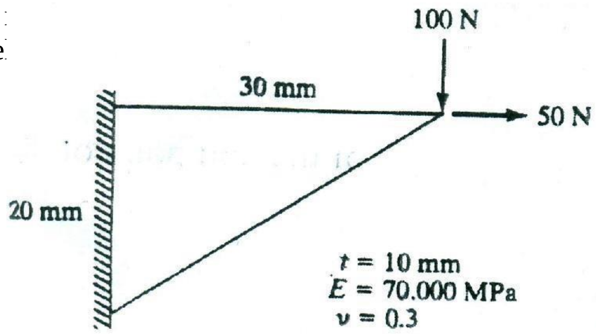
4. a) Write the difference between CST and LST elements [3M]

b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P. [7M]



OR

5 For the configuration shown in Fig. determine the deflection at the point of load application using a one-element : used, comment on the stress values in the e [10M]



S

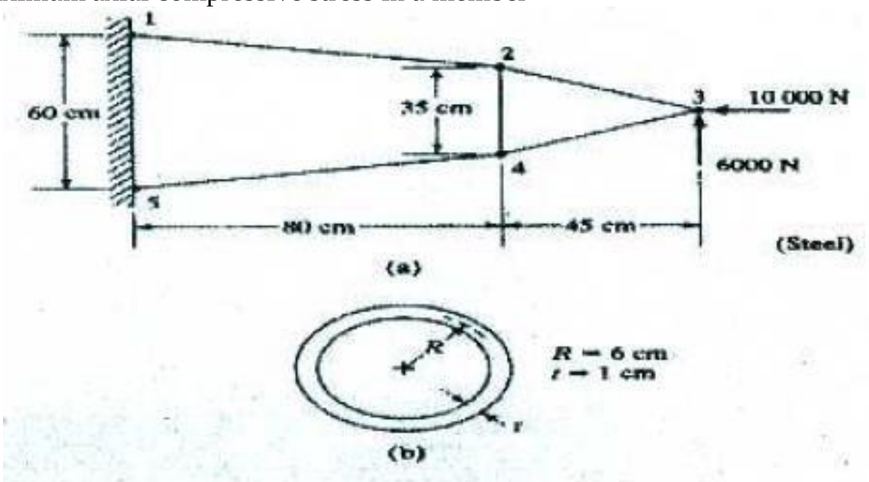
6. Derive the strain displacement matrix for axisymmetric triangular element Discuss advantages of axisymmetric modelling in FEM [10M]

OR

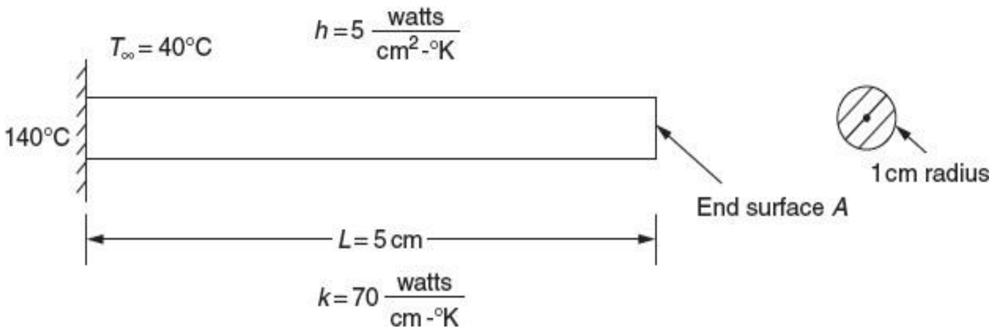
7. Figure shows a five – member steel frame subjected to loads at the free end. The cross section of each member is a tube of wall thickness t=1 cm and mean radius=6cm. Determine the following: [10M]

a) The displacement of node 3 and

b) The maximum axial compressive stress in a member



8. Find the temperature distribution in the one-dimensional fin shown in Figure below using two finite elements. [10M]



OR

9. (a) A 20-cm thick wall of an industrial furnace is constructed using fireclay bricks that have a thermal conductivity of k = 2 W/m-°C. During steady state operation, the furnace wall has a temperature of 800°C on the inside and 300°C on the outside. If oneof the walls of the furnace has a surface area of 2 m^2 (with 20-cm thickness), find the rate of heat transfer and rate of heat loss through the wall. [5M]

(b) A metal pipe of 10-cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20°C while the outer surface of the pipe is at a temperature of 250°C. If the heat transfer coefficient for free convection from the pipe to the air is h = 20 W/m^2-°C find the rate of heat loss from the pipe. [5M]

10. For the two-bar truss shown in Figure below, determine the nodal displacements, element stresses and support reactions. A force of P=1000kN is applied at node-1. Assume E=210GPa and A=600mm2 for each element. [10M]

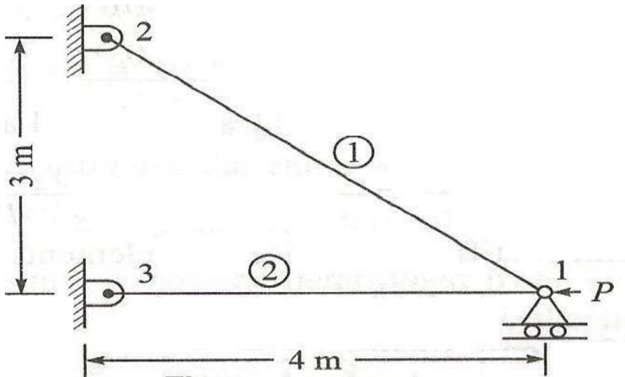


Figure two-bar truss.

OR

11. A bar of length 1 m; cross sectional area 100 mm2; density of 7 gm/cc and Young's modulus 200Gpa is fixed at both the ends. Consider the bar as three bar elements and determine the first two natural frequencies and the corresponding mode shapes. Discuss on the accuracy of the obtained solution [10M]

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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, December 2019

Finite Element Method

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B

Part A is compulsory which carriers 25 marks and Answer all questions.

Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

****

PART-A (25 Marks)

1). a	What is meant by Engineering analysis and specify its Types	**[2M]**
b	Explain finite element method?	**[3M]**
c	Draw a plane truss structure.	**[2M]**
d	What are the characteristics of a truss?	**[3M]**
e	Define shape function.	**[2M]**
f	List any four two dimensional elements.	**[3M]**
g	What is Fourier's law?	**[2M]**
h	Discuss the types of heat transfer	**[3M]**
i	What is consistent mass matrix?	**[2M]**
j	Define Eigen values?	**[3M]**

PART-B (50 MARKS)

SECTION-I

2 Explain the concept of FEM briefly and outline the steps involved in FEM along with remembers. **[10M]**

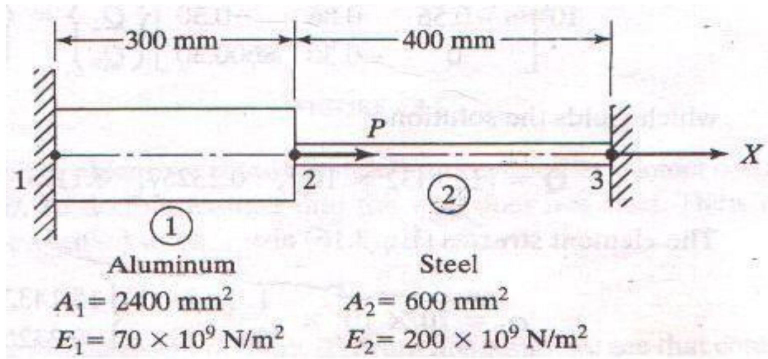
OR

3 Consider the following fig. An axial load P=200 KN is applied as shown. Using penalty approach for handling boundary conditions, do the following **[10M]**

a) Determine the nodal displacements.

b) Determine the stress in each material.

c) Determine the reaction forces.


300 mm
400 mm
P
X
1
2
3
1
2
Aluminum
Steel
A1= 2400 mm2
E1= 70 × 10^9 N/m2
A2= 600 mm2
E2= 200 × 10^9 N/m2

SECTION-II

4 For the two-bar truss shown in Figure below, determine the nodal displacements, element stresses and support reactions. A force of P=1000kN is applied at node-1. Assume E=210GPa and A=600mm^2 for each element. **[10M]**

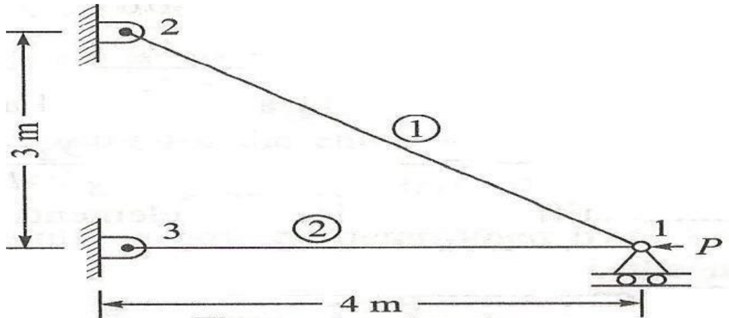


Figure two-bar truss.

OR

5 a).Explain Iso-parametric, sub-parametricand super- parametric elements **[6M]**

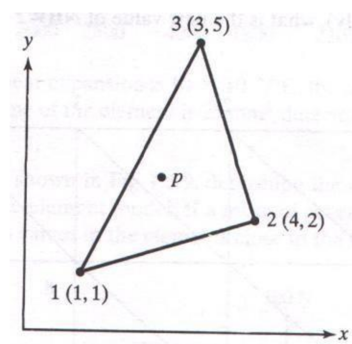
b) Advantages of iso-parametric elements **[4M]**

SECTION-III

6 Explain the concept of numerical integration and its utility in generating Isoperimetric finite element matrices. **[10M]**

OR

7 For the point P located inside the triangle, the shape functions N_1 and N_2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of P. **[10M]**

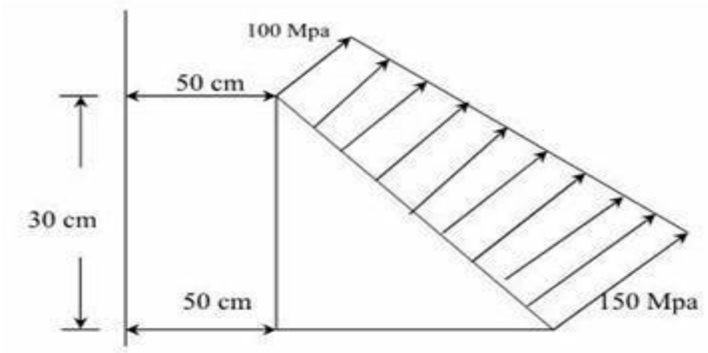


SECTION-IV

8 Estimate the temperature profile in **a** fin of diameter 25 mm, whose length is 500mm. The thermal conductivity of the fin material is 50 W/m K and heat transfer coefficient over the surface of the fin is 40 W/m^2 K at 30^{0}C. The tip is insulated and the base is exposed to a temperature of 150 ^{0}C. Evaluate the temperatures at points separated by 100 mm each. **[10M]**

OR

9 An axi-symmetric triangular element is subjected to the loading as shown in fig. the load is distributed throughout the circumference and normal to the boundary. Derive all the necessary equations and derive the nodal point loads. **[10M]**



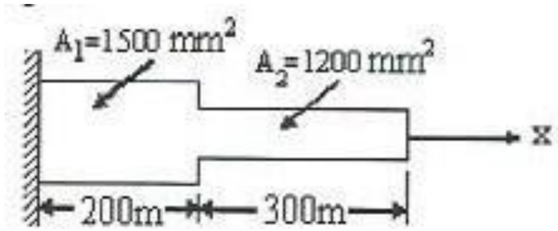
SECTION-V

10 Explain the following with examples: **[10M]**

a) Lumped mass matrix. b) Types of vibrations.

OR

11 Determine the natural frequencies and mode shapes of a stepped bar shown in figure below. Assume E=300GPa and density is 7800 Kg/m^3. **[10M]**



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